Neutral Equations of Mixed Type

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Abstract

In this dissertation we consider neutral equations of mixed type. In particular, we consider the associated linear Fredholm theory and nerve fiber models that are written as systems of neutral equations of mixed type.

In Chapter 2, we extend the existing Fredholm theory for mixed type functional differential equations developed by Mallet-Paret [31] to the case of implicitly defined mixed type functional differential equations.

In Chapter 3, we apply the theory to an example arising from modeling signal propagation in nerve fibers. In this two-dimensional system, we rely on the Lyapunov-Schmidt method to demonstrate the existence of traveling wave solutions. With the aid of numerical computations, a saddle-node bifurcation was detected.

In Chapter 4, we consider an extension of the parallel nerve fiber model examining in Chapter 3 and present the results of a numerical study. In this chapter, an additional form of coupling is examined not considered in the model from Chapter 3. This second type of coupling may be excitatory or inhibitory depending on the sign of the coupling parameter. Within a continuation framework, we employ a pseudo-spectral approach utilizing Chebyshev polynomials as basis functions. The `chebfun` package [7], consisting of Chebyshev tools, was utilized to manipulate the polynomials.

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Contents

Chapter 1

Introduction

A gradual introduction to neutral equations of mixed type begins with a discussion of delay differential equations. Modeling with delay differential equations incorporates information not only at the present time t but also τ units prior at time $t-\tau$. Delay differential equations have been commonly used to model population dynamics. Hutchinson's equation, also called the delayed logistic equation,

$$\dot{x}(t) = rx(t)\left[1 - \frac{x(t-\tau)}{K}\right],$$

is an example of a delay differential equation used to model population dynamics. The parameters r and K are standard parameters found in population models representing the growth rate and carrying capacity, respectively. Our interest lies in the delay parameter τ. By incorporating the delay parameter, Hutchinson's model is able to accommodate an additional level of detail in the form of an incubation time required for eggs to mature and hatch. In general, delay equations and other functional differential equations may have close parallels with more familar ode's and pde's, but an additional level of detail may be gained by studying the less common delay equation or functional differential equation.

More generally, delay and neutral equations fall under the larger classification of functional differential equations. A functional differential equation is any differential equation where the unknown function is evaluated at different argument values, e.g. t and $t-\tau$. *Neutral* differential equations are a class of equations involving derivatives that may contain shifted arguments of the independent variable. Lastly, the term *mixed type* refers to problems containing both forward (advances) and backward (delays) shifts of the independent variable.

This dissertation will examine neutral equations of mixed type. However, before proceeding any further, some clarification is needed in the classification of neutral equations of mixed type. Consider the explicitly defined functional differential equation:

$$x'(t) = Ax(t+\sigma) + Bx(t) + Cx(t+\tau).$$

If $\sigma > 0$ and $\tau < 0$, then the shifts are of opposite sign, and the equation is of mixed type in relation to the argument of the derivative term. However, the classification of neutral equations of mixed type requires more consideration. The following neutral equation is not of mixed type:

$$Dx'(t+\sigma) + Ex'(t) + Fx'(t+\tau) = Ax(t+\sigma) + Bx(t) + Cx(t+\tau).$$

If the change of variables $\xi = t+\sigma$ is made, the transformed equation

$$Dx'(\xi) + Ex'(\xi-\sigma) + Fx'(\xi+\tau-\sigma) = Ax(\xi) + Bx(\xi-\sigma) + Cx(\xi+\tau-\sigma)$$

is a neutral delay equation if the leading coefficients D and A are nonzero. The following equation is of mixed type:

$$Dx'(t+\sigma)+Ex'(t)+Fx'(t-\sigma)=Ax(t+\tau)+Bx(t)+Cx(t+\tau),$$

where $\tau>\sigma>0$. Regardless of making the change of variables $\xi=t+\sigma$, the right-hand side remains of mixed type. As an operational definition, we view the neutral model of mixed type in the form where the derivative term with the largest coefficient is centered with no shift. In fact, in the Ephaptic model later considered, the left-hand side is a perturbation of the identity.

It is worth noting that neutral models will in general have discontinuities in the derivative of the solution. An example of the lack of smoothness comes from examining the initial value problem for a neutral delay problem mentioned in Hale, Verduyn Lunel [23]:

$$x'(t)-Cx'(t-\tau)=Ax'(t)+Bx(t-\tau)+f(t)$$

with the given initial data $x(\xi)=\phi(\xi)$ on $[-\tau,0]$, where ϕ is continuously differentiable. If

$$\phi'(0)\neq C\phi'(-\tau)+A\phi(0)+B\phi(-\tau)+f(0),$$

then x' is discontinuous at $t=0$ and since $C\neq 0$ the discontinuities in x' are propagated along at $t=k\tau, k=1,2,...$ If additional shifts are incorporated into the neutral model the quantity and nature of the shifts become more complex.

In the hereafter, we will not consider initial value problems, but boundary value problems. Our intention is to find traveling wave solutions (ϕ,c) to differential equations posed on a lattice, $n \in \mathbb{Z}$. To accomplish this goal, we will utilize continuation techniques. Suppose the equation under consideration may be rewritten in the form of a zero finding problem:

$$\mathscr{G}(\phi, c, \alpha) = 0.$$

Once the equation is in this form, Newton's method can be applied.

The application of Newton's method is predicated upon the derivative (linearization) of $\mathscr{G}$ with respect to the traveling wave $x = (\phi, c)$ being invertible

$$x^{k+1} = x^k - [D_x \mathscr{G}(x^k, \alpha)]^{-1} \mathscr{G}(x^k, \alpha). \tag{1.1}$$

The linear Fredholm theory developed in the sequel will address the invertibility of the operator $D_x \mathscr{G}$. We extend the existing theory to the neutral case in Chapter 2, and under a certain set of assumptions the linear operator $D_x \mathscr{G}$ may be viewed as a Fredholm operator which means its invertibility may be determined by examining its Fredholm index and the dimension of the kernel of the linear operator.

Following the linear theory, two neutral mixed type models are examined. In particular, traveling waves are shown to exist for a two-dimensional system which, in short, is termed the Ephaptic model in Chapter 3. A local continuation strategy is employed whereby the initial solution is known when the coupling parameter takes the value $\alpha = 0$. In fact, the system completely decouples when $\alpha = 0$, and two copies of the same wave profile with a common wave speed solve the two uncoupled equations. However, when the coupling is turned on, two arbitrary copies of the same wave profile do not solve the coupled system. The Lyapunov-Schmidt method is applied to the system to give a reduction to a finite dimensional bifurcation equation. It is shown via the Lyapunov-Schmidt method that the system has solutions, and moreover, undergoes a saddle-node bifurcation.

Chapter 4 builds upon Chapter 3 by utilizing path-following techniques to compute solutions to a more general version of the Ephaptic system. This model contains an additional form of coupling which has been considered in related continuum models by Bose [12]. We utilize a pseudo-spectral approach with Chebyshev basis functions.

Chapter 2

Neutral Equations of Mixed Type

2.1 Introduction

In several areas of science and engineering modeling results in differential equations that are implicitly defined. For instance, the finite element method when applied to parabolic systems yields an implicitly defined equation with an associated mass matrix B. In control theory, implicitly defined differential algebraic equations can be encountered after applying Kirchoff's laws. Our interest here is in implicitly defined functional differential equations of mixed type, i.e. with both advances and delays. The types of problems we consider here may be thought of as the analogue of neutral delay equations for problems with advances and delays.

Our contribution in this paper is to extend the existing Fredholm theory of Mallet-Paret [31] for explicitly defined mixed type functional differential equations to account for implicitly defined equations, and we also will examine two neutral mixed type equations in detail. In particular, we will demonstrate the existence of traveling wave solutions via local continuation to a system of coupled discrete Nagumo equations with a bistable nonlinearity. Our motivation is to develop a linear theory for problems with implicit coupling and to provide an alternative means for modeling problems such as those considered by Bates, Chen, and Chmaj [6] with infinite range interaction. In addition, the linear theory is applicable to both neutral delay equations and problems not well-posed as IVPs.

Our interest in implicitly defined equations of mixed type arose from examining electrical signaling in cardiac tissue and nerve conduction models. Ultimately, we are looking for traveling wave solutions (ϕ, c) with waveform ϕ and wave speed c that satisfy the following equation

$$\sum_{j=1}^{N} B_j(\xi)[-c\phi'(\xi + r_j) + f(\phi(\xi + r_j))] = \sum_{j=1}^{N} A_i(\xi)\phi(\xi + r_j) \tag{2.1}$$

with boundary conditions $\phi(-\infty) = \phi_-, \phi(\infty) = \phi_+$. Note, as stated, Equation (2.1) is implicitly defined, where the explicit case occurs when $B_1(\xi) = I$, and $B_j(\xi) = 0$ for $j \in \{2, \ldots, N\}$.

Also, note that an alternative to solving for a closed form system of implicitly defined differential equations is to leave the equations in a differential-algebraic form, which is a standard form [13] in circuit analysis. For example, in our analysis of the nerve fiber model considered in Chapter 3, an application of Kirchoff's Voltage and Current laws result in algebraic and differential equations, respectively. In addition, circuits modeled with certain types of components called unicursal elements (see Chapter 3 of [13]) can give rise to systems with a nonlinearity on the left-hand side

$$A(x)\dot{x} = B(x) + f(t). \tag{2.2}$$

While these types of circuits are outside the scope of this paper they further demonstrate the occurrence of models which cannot be posed explicitly.

The continuation procedure for solving (2.1) involves employing an appropriate Fredholm theory. Ultimately, a known solution "close" (differing by a small perturbation) to the desired problem (2.1) is needed as a starting point for continuation. By linearizing about the equilibria of the starting problem, one can effectively utilize the linear Fredholm theory to continue solutions along a homotopy path. Under suitable assumptions the Fredholm index remains zero along the entire path linking the starting problem with the desired problem, thus allowing continuation between problems. However, we will perform local continuation not global continuation by applying either the implicit function theorem or by utilizing the Lyapunov-Schmidt method.

In this paper, the Fredholm theory for implicitly defined functional differential equations of mixed type will extend the explicit case considered by Mallet-Paret [31]. If equation (2.1) is posed as

$$B\phi' = A\phi + F(\phi) \tag{2.3}$$

where $B, A,$ and F are infinite dimensional matrices, then the equation could be reduced to an explicit version by inverting B, assuming B is invertible, and potentially arrive at an equation with infinite range interaction. Due to the potential fill-in present in B^{-1} we seek to directly work with the implicit form.

The study of the Fredholm theory for linear non-autonomous ordinary differential equations is shown by Palmer [35] to be closely related to exponential dichotomies. In essence, as long as the system $\dot{x} = A(t)x$ exhibits an exponential dichotomy on both half lines, then the linear operator L, $Lx = \dot{x} - A(\bullet)x$, is Fredholm. Beyn [10] utilized the Fredholm theory to develop the underpinnings for the numerical computation of connecting orbits. Lin [30] extended the Fredholm theory to delay differential equations. The papers [24] and [34] discuss exponential dichotomies in mixed type problems.

The study of the theory of functional differential equations of mixed type began with Rustichini [39]. In particular, Rustichini identifies some of the key properties of mixed type problems. In a companion paper [40], Rustichini proves a Hopf bifurcation theorem and subsequently applies it to study the Hopf bifurcation arising from a system governing an economic model.

Functional differential equations of mixed type can arise from systems of ordinary differential equations modeled on a spatial lattice. In particular, Mallet-Paret has a survey paper [33] discussing the dynamics of systems posed on a spatial lattice. Mallet-Paret's companion papers [31] and [32] develop the Fredholm theory for mixed type problems as well as study the global structure of the set of traveling wave solutions. Hupkes and Verduyn Lunel have developed a center manifold theory [25] and have adapted Lin's method [27] for mixed type problems.

In Section 2 we develop our notation for the implicitly defined case. In Section 3 we will obtain the Green's function for the constant coefficient case. In Section 4 we will develop the Fredholm alternative theory for the neutral/implicitly defined case. In Section 5 we will develop the cocycle property and the spectral flow property. Finally, in Sections 2.6-3.1 we will apply our theory to some examples to demonstrate its usage.

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2.2 Background

2.2.1 Notation

The form of the neutral/implicitly defined linear functional differential equation under consideration is

$$\sum_{j=1}^{N} B_j(\xi)x'(\xi+r_j) = \sum_{j=1}^{N} A_j(\xi)x(\xi+r_j)+h(\xi), \tag{2.4}$$

where the shifts r_j can take either sign and follow the convention set forth in [31]

$$r_1 = 0, \text{ and } r_j \neq r_k \text{ for } j \neq k. \tag{2.5}$$

In addition, the ordering of the remaining shifts begin with the smallest magnitude shift and ending with the largest magnitude shift. If shifts of equal magnitude are present, the negative shift will be labeled first. The largest forward shift and the largest backward shift are labeled as follows

$$r_{\min} = \min_{1\leq j\leq N}\{r_j\}, \text{ and } \qquad r_{\max} = \max_{1\leq j\leq N}\{r_j\}. \tag{2.6}$$

We assume the coefficients B_j, A_j are measurable and uniformly bounded on the real line, and note that the coefficients may in fact vanish identically on $\mathbb{R}$.

Maintaining consistency with [31], we can define the state of the solution $x_\xi \in C([r_{\min}, r_{\max}], C^d)$ by $x_\xi(\theta) = x(\xi+\theta)$ for $\theta \in [r_{\min}, r_{\max}]$. As a shorthand Eq. (2.4) can be rewritten as

$$D(\xi)x'_\xi = L(\xi)x_\xi + h(\xi), \tag{2.7}$$

where the linear operators $D : L^p([r_{\min}, r_{\max}], \mathbb{C}^d) \to \mathbb{C}^d, L : C([r_{\min}, r_{\max}], \mathbb{C}^d) \to \mathbb{C}^d$ are defined for almost every $\xi \in \mathbb{R}$ as

$$D(\xi)x'_{\xi} = \sum_{j=1}^{N} B_j(\xi)x'(\xi + r_j), \quad x' \in L^p([r_{\min}, r_{\max}], \mathbb{C}^d), \tag{2.8}$$

$$L(\xi)x_{\xi} = \sum_{j=1}^{N} A_j(\xi)x(\xi + r_j), \quad x \in C([r_{\min}, r_{\max}], \mathbb{C}^d). \tag{2.9}$$

With the shorthand $L^p \equiv L^p(\mathbb{R}, \mathbb{C}^d)$, we introduce the following Sobolev spaces

$$W^{1,p} = \{f \in L^p \mid f \text{ is absolutely continuous and } f' \in L^p\}, \text{ and}$$
$$W^{1,p}_0 = \{f \in W^{1,p} \mid f(0) = 0\}.$$

Letting $\Lambda_{D,L} y = D(\xi)y'_{\xi} - L(\xi)y_{\xi}$, we note that $\Lambda_{D,L} : W^{1,p} \to L^p$. In the homogeneous case, (2.4) reduces to

$$D(\xi)x'_{\xi} = L(\xi)x_{\xi}. \tag{2.10}$$

In the case that the coefficient matrices $B_j(\xi), A_j(\xi)$ are constant, we have the constant coefficient linear operators

$$D_0 x'_{\xi} = \sum_{j=1}^{N} B_{j,0}x(\xi + r_j), \quad L_0 x_{\xi} = \sum_{j=1}^{N} A_{j,0}x(\xi + r_j), \tag{2.11}$$

where $B_{j,0}, A_{j,0}$ are constant coefficient matrices. In the constant coefficient case, (2.4) takes the form

$$D_0 x'_{\xi} = L_0 x_{\xi} + h(\xi), \tag{2.12}$$

with the homogeneous equation being

$$D_0 x'_{\xi} = L_0 x_{\xi}, \tag{2.13}$$

with the operator $(\Lambda_{D_0,L_0}x)(\xi) = D_0 x'_\xi - L_0 x_\xi$. The adjoint equation associated to (2.10) is given by

$$D^*(\xi)y'_\xi = L^*(\xi)y_\xi \tag{2.14}$$

where

$$\begin{aligned} D^*(\xi)y'_\xi &= -\textstyle\sum_{j=1}^N B_j(\xi - r_j)^* y'(\xi - r_j), \\ L^*(\xi)y_\xi &= \textstyle\sum_{j=1}^N A_j(\xi - r_j)^* y(\xi - r_j). \end{aligned} \tag{2.15}$$

Let $\Lambda^*_{D,L}y = D^*(\xi)y'_\xi - L^*(\xi)y_\xi$. We note the adjoint operator is the map $\Lambda^*_{D,L} : W^{1,q} \to L^q$, where $1/p + 1/q = 1$, and so that the integral in the Fredholm Alternative (2.92) makes sense.

We can decompose each of the operators $D(\xi)$ and $L(\xi)$ into a sum of a constant coefficient operator and a variable coefficient operator

$$D(\xi) = D_- + M_{1,-}(\xi) = D_+ + M_{1,+}(\xi), \tag{2.16}$$

$$L(\xi) = L_- + M_{2,-}(\xi) = L_+ + M_{2,+}(\xi). \tag{2.17}$$

Definition 2.2.1. *The system (2.10) is asymptotically autonomous at $\pm\infty$ if there exists $D_-, D_+, M_{1,-}$, and $M_{1,+}$ with the limits*

$$\lim_{\xi\to-\infty} ||M_{1,-}(\xi)|| = \lim_{\xi\to\infty} ||M_{1,+}(\xi)|| = 0, \tag{2.18}$$

and if similar operators exist for L.

Note, if the system (2.10) is indeed asymptotically hyperbolic, then we emphasize that the limiting equations at $\pm\infty$ need not be the same, e.g. $L_- \neq L_+$.

The characteristic matrix associated with the constant coefficient system (2.13) is found by substituting the ansatz $x(\xi) = e^{\lambda\xi}$, and subsequently dividing the resulting expression by $e^{\lambda\xi}$

$$\Delta_{D_0,L_0}(\lambda) = \lambda \sum_{j=1}^{N} B_{j,0} e^{\lambda r_j} - \sum_{j=1}^{N} A_{j,0} e^{\lambda r_j}. \tag{2.19}$$

The characteristic equation is then $det(\Delta_{D_0,L_0}(\lambda)) = 0$.

Definition 2.2.2. *The constant coefficient system (2.13) is hyperbolic if*

$$det(\Delta_{D_0,L_0}(i\eta)) \neq 0. \qquad \forall \eta \in \mathbb{R} \tag{2.20}$$

Moreover, if the system (2.10) is asymptotically autonomous and the limiting equations at $\pm\infty$ are hyperbolic then we say (2.10) is asymptotically hyperbolic.

2.2.2 Fourier Analysis

The Schwartz space $\mathscr{S}$ consists of C^∞ functions decaying faster than the reciprocal of any polynomial. For any function $f \in \mathscr{S}$ the form of the Fourier transform is given by

$$\hat{f} = \mathscr{F}[f](\eta) = \int_{-\infty}^{\infty} e^{-i\xi\eta} f(\xi) d\xi, \tag{2.21}$$

and the corresponding inverse transform is given by

$$\check{g} = \mathscr{F}^{-1}[g](\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi\eta} g(\eta) d\eta. \tag{2.22}$$

Continuous linear functionals on the Schwartz functions are called tempered distributions, denoted by $\mathscr{S}'$. To apply Fourier transforms to L^p functions for $p \neq 1$ it is necessary to view each L^p function as a tempered distribution.

Let D and τ_{r_j} denote the derivative and translation operators, respectively, i.e. $D = {}'$ and $(\tau_{r_j}x)(\xi) = x(\xi + r_j)$. Recall, the Fourier transform of the derivative of a function can be identified with the multiplier $i\eta$. For $\phi \in \mathscr{S}$

$$< \widehat{Dx}, \phi > = < Dx, \hat{\phi} > = - < x, D\hat{\phi} > = - < x, \widehat{-i\eta\phi} > = < i\eta\hat{x}, \phi > .$$

In addition, the Fourier transform of the translation operator can be identified with the multiplier $e^{i\eta r_j}$

$$< \widehat{\tau_{r_j}x}, \phi > = < \tau_{r_j}x, \hat{\phi} > = < x, \tau_{-r_j}\hat{\phi} > = < x, \widehat{e^{i\eta r_j}\phi} > = < e^{i\eta r_j}\hat{x}, \phi > .$$

As a result, the Fourier transform of the left-hand side operator in (2.13) can be identified with the following

$$\begin{aligned} \mathscr{F}[D_0 x'] &= \mathscr{F}[\sum_{j=1}^{N} B_{j,0} x'(\xi + r_j)] \\ &= \sum_{j=1}^{N} \mathscr{F}[B_{j,0} D\tau_{r_j} x] = i\eta(\sum_{j=1}^{N} B_{j,0} e^{i\eta r_j})\hat{x}(\eta). \end{aligned} \tag{2.23}$$

2.2.3 Matrix-Valued Functions

We define the matrix-valued function $B(s) : \mathbb{C} \to \mathbb{C}^{d\times d}$ as

$$B(s) = \sum_{j=1}^{N} B_{j,0} e^{sr_j}, \tag{2.24}$$

where $B(i\eta)$ is the multiplier identified with the Fourier transform of D_0 as in (2.23). Similarly, let $A(s) : \mathbb{C} \to \mathbb{C}^{d\times d}$ be defined as

$$A(s) = \sum_{j=1}^{N} A_{j,0} e^{sr_j}. \tag{2.25}$$

The periodicity of the functions $B(i\eta)$ and $A(i\eta)$ depend on the interrelationship of the shifts r_j (see Bellman and Cooke [8] for reference). The shifts are rationally related if there exists a number ν such that $r_j = d_j\nu$, where d_j are integers. Let ν be the minimal number given from satisfying the condition $\gcd(d_2, d_3, \ldots, d_N) = 1$.

Proposition 2.2.3. *The periodicity of the functions $B(i\eta), A(i\eta)$ is given by $p = 2\pi\nu^{-1}$. When no ν exists the shifts are not rationally related, and the functions $B(i\eta)$ and $A(i\eta)$ are aperiodic.*

For instance, the function $B(i\eta)$ is aperiodic when $r_1 = 0, r_2 = -1$, and $r_3 = \sqrt{2}$.

In the rationally related case, the matrix-valued function $B^{-1}(i\eta)$ is also p periodic. Assuming $B^{-1}(s)$ is invertible in a vertical strip S about the imaginary axis, then $B^{-1}(s)$ is a matrix-valued function with holomorphic entries thereby permitting a Fourier series representation on S, see [38] for reference:

Theorem 2.2.4. *Let f be holomorphic and p periodic in the strip S. Then f can be expanded into a unique Fourier series*

$$f(z) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{2\pi i k}{p} z} \tag{2.26}$$

which is normally convergent to f in S.

Recall, a normally convergent series $\sum_n f_n$ in S requires

$$\sum_n \sup_S |f_n(z)| < \infty. \tag{2.27}$$

To ensure that the characteristic matrix $\Delta_{D_0,L_0}(s)$ is invertible in a strip about the imaginary axis, conditions on $B(s)$ need to be imposed. By writing the characteristic matrix $\Delta_{D_0,L_0}(s)$ in the following form

$$\Delta_{D_0,L_0}(s) = sB(s) - A(s), \tag{2.28}$$

one can see that within any vertical strip $\mathscr{S} = \{s \big| |\text{Re } s| \leq a_0\}$, in which $B(s)$ has a bounded inverse, the following relationship holds

$$B^{-1}(s)\Delta_{D_0,L_0}(s) = sI - B^{-1}(s)A(s) = sI + O(1), \text{ as } |\text{Im } s| \to \infty. \tag{2.29}$$

Since B^{-1} and A are bounded in the strip $\mathscr{S}$, the roots of the right-hand side are confined to a compact interval in which $|s| \leq ||B^{-1}A||_{L^\infty}$. Since analytic functions have isolated roots, the right-hand side has a finite number of roots within $\mathscr{S}$, implying $\Delta_{D_0,L_0}(s)$ has a finite number of roots within $\mathscr{S}$. As a result there exists a constant $\tilde{a}_0 > 0$ and a vertical strip $\tilde{\mathscr{S}} = \{s \big| |\text{Re } s| \leq \tilde{a}_0\}$ in which $\Delta_{D_0,L_0}(s)$ is invertible. When considering left-hand side difference operators D_0 and their associated matrix-valued functions $B(s)$, we have the following assumption in mind

Assumption A. *$B(s)$ has a bounded inverse on the imaginary axis.*

As in [8], the determinant of $B(s)$ takes the following form

$$g(s) = \det(B(s)) = \sum_{i=0}^{n} p_j e^{\beta_j s}, \tag{2.30}$$

where the β_j's are linear combinations of the shifts r_j's, and the coefficients p_j's are constants. If $B(s)$ has a bounded inverse by assumption, then $|g(s)| \geq c > 0$. Consequently, $B(s)$ is invertible in a strip about the imaginary axis due to the uniform continuity of $g(s)$ in vertical strips.

We note, Assumption A can be satisfied. The condition of block strictly diagonally dominance is sufficient, if assumed, to guarantee that $B(s)$ satisfies Assumption A. If the matrices $B_{j,0}$, $j = 1...N$, are viewed as block elements of a larger matrix B, where $B_{1,0}$ is the diagonal element, then the following definition can be made as in [21].

Definition 2.2.5. *The function $B(s)$ is* ***block strictly diagonally dominant*** *if $B_{1,0}$ is invertible, and the following inequality is satisfied*

$$|B_{1,0}^{-1}|^{-1} > \sum_{j=2}^{N} |B_{j,0}|. \tag{2.31}$$

A block strictly diagonally dominant $B(s)$ assumes an invertible $B_{1,0}$, permitting the rewriting

$$B(s) = B_{1,0}(I + B_{1,0}^{-1} \sum_{j=2}^{N} B_{j,0} e^{sr_j}).$$

It follows that the quantity $(I + B_{1,0}^{-1} \sum_{j=2}^{N} B_{j,0} e^{i\eta r_j})$ is invertible since

$$|B_{1,0}^{-1}| \sum_{j=2}^{N} |B_{j,0}| < 1.$$

This statement guarantees that $B(s)$ has a bounded inverse on the imaginary axis. In addition, if we set $s = a + bi$, the following inequality holds

$$|B_{1,0}^{-1} \sum_{j=2}^{N} B_{j,0} e^{(a+bi)r_j}| \leq |B_{1,0}^{-1}| \sum_{j=2}^{N} |B_{j,0}| e^{ar_j} \leq |B_{1,0}^{-1}| \max_j e^{ar_j} \sum_{j=2}^{N} |B_{j,0}|.$$

By continuity a strip $\mathscr{S} = \{s \big| |\text{Re } s| \leq \tilde{a}_0\}$ exists about the imaginary axis where $B(s)$ maintains block strictly diagonal dominance and hence invertibility. In addition, the inverse $B^{-1}(s)$ is uniformly bounded in the vertical strip $\mathscr{S}$.

2.2.4 Exponential Dichotomy for Difference Equations

We find that Assumption A has an equivalent notion in terms of exponential dichotomy. This equivalence leads to a discussion about exponential dichotomy. We present the following definition about exponential dichotomies for difference equations found in [36] (see also, e.g., [3]).

Definition 2.2.6. *For $k \in J$, an interval in $\mathbb{Z}$, let A_k be an invertible $n \times n$ matrix. The difference equation*

$$u_{k+1} = A_k u_k \tag{2.32}$$

*is said to have an **exponential dichotomy** on J if there are projections P_k and positive constants $K_1, K_2, \lambda_1, \lambda_2$ with $\lambda_1 < 1, \lambda_2 < 1$ such that for $k, m \in J$ the projections satisfy the invariance conditions*

$$\Phi(k,m)P_m = P_k \Phi(k,m), \tag{2.33}$$

and the inequalities

$$|\Phi(k,m)P_m| \leq K_1 \lambda_1^{k-m}, \quad k \geq m \tag{2.34}$$

and

$$|\Phi(k,m)(I-P_m)| \leq K_2 \lambda_2^{m-k}, \quad k \leq m \tag{2.35}$$

hold. Where $\Phi(k,m)$ *is the transition matrix defined by*

$$\Phi(k,m) = \begin{cases} A_{k-1}\ldots A_m, & \text{for } k > m, \\ I, & \text{for } k = m, \\ \Phi(m,k)^{-1}, & \text{for } k < m. \end{cases} \tag{2.36}$$

In the case that A_k is a constant matrix A with eigenvalues strictly greater than 1 and/or strictly less than 1, then the transition matrix reduces to powers of A

$$\Phi(k,m) = A^{k-m}.$$

The projections P_k reduce to constant P, and the difference equation (2.32) exhibits an exponential dichotomy on the entire real line.

Proposition 2.2.7. *Suppose that* $u_{k+1} = Au_k$ *exhibits an exponential dichotomy on* $\mathbb{Z}$ *with a projection* P *and constants* $K_1, K_2, \lambda_1, \lambda_2$ *given by (2.33)-(2.35). Then, for any* $f \in l^p$, $1 \leq p \leq \infty$, *the inhomogeneous problem*

$$u_{k+1} = Au_k + f_k \tag{2.37}$$

has a solution $u \in l^p$. *Moreover, if* f_k *exhibits exponential decay of the form* $|f_k| \leq Ce^{-\alpha|k|}$, *then* u_k *also exhibits exponential decay of the following form:*

$$|u_k| \leq \tilde{C}e^{-\min\{\alpha,\beta\}|k|}, \tag{2.38}$$

where $-\beta = \ln(\lambda)$ *and* $\lambda = \max\{\lambda_1, \lambda_2\}$.

Proof. We begin with the following representation of u_k:

$$u_k = \sum_{l=-\infty}^{k} A^{k-l} P f_l - \sum_{l=k+1}^{\infty} A^{k-l} (I-P) f_l \tag{2.39}$$

which gives the following inequality:

$$|u_k| \leq K_1 \sum_{l=-\infty}^{k} \lambda_1^{k-l} |f_l| + K_2 \sum_{l=k+1}^{\infty} \lambda_2^{k-l} |f_l|. \tag{2.40}$$

Let $K = \max\{K_1, K_2\}$ and $\lambda = \max\{\lambda_1, \lambda_2\}$, then

$$|u_k| \leq K \sum_{l=-\infty}^{\infty} \lambda^{|k-l|} |f_l|. \tag{2.41}$$

If $1 \leq p < \infty$, the following bound holds:

$$\sum_k |u_k|^p \leq K^p \sum_k \left(\sum_{l=-\infty}^{\infty} \lambda^{|k-l|} |f_l| \right)^p. \tag{2.42}$$

Using Young's inequality for convolutions we have $a_k = \sum_{l=-\infty}^{\infty} \lambda^{|k-l|} |f_l| \in l^p$ if $\{\lambda^{|k|}\} \in l^1$ and $f_l \in l^P$. Since $\sum_k \lambda^{|k|}$ is a geometric series with $|\lambda| < 1$, it follows that $\{\lambda^{|k|}\} \in l^1$ implying that $u_k \in l^p$. Similarly by Young's inequality, if $p = \infty$, $u_k \in l^\infty$ since $\{\lambda^{|k|}\} \in l^1$.

To demonstrate that u_k exhibits exponential decay, we build upon (2.41):

$$\begin{aligned} |u_k| &\leq K \sum_{l=-\infty}^{\infty} \lambda^{|k-l|} |f_l| \leq CK \sum_{l=-\infty}^{\infty} \lambda^{|k-l|} e^{-\alpha|t|} \leq CK \sum_{l=-\infty}^{\infty} \int_l^{l+1} e^{-\beta|k-t|} e^{-\alpha|t|} dt \\ &\leq CK \int_{-\infty}^{\infty} e^{-\beta|k-t|} e^{-\alpha|t|} dt \leq \tilde{C} e^{-\max\{\alpha,\beta\}|k|}. \end{aligned} \tag{2.43}$$

∎

To view the difference equation associated with the left-hand side of Eq. (2.4), let $y(\xi) = x'(\xi)$ as in [23], and fix a value $\xi_0 \in [0, \nu)$. Note, the size of the system is a product of the periodicity p. When $p = 2\pi\nu^{-1}$ is large the size of associated difference equation is large, and in particular, y needs to be evaluated every ν units. As a result, the shifts r_j are integer multiples of ν units apart

$$r_k = r_j + (d_k - d_j)\nu, \tag{2.44}$$

and the size of the system of difference equations is $M = (d_{max} - d_{min})$. By labeling the points $t_m = \xi_0 + r_{min} + m\nu$ for $0 \leq m \leq M-1$ the vector u_k from (2.37) can be viewed as $u_k^T = [y(t_0 + k\nu), y(t_1 + k\nu), \ldots, y(t_{M-1} + k\nu)]^T$. We caution that care must be taken in the case of a non-invertible leading coefficient B_N. In this case, the order of the system of difference equations needs to be suitable reduced, see Section 3.0.11.

Lemma 2.2.8. *Let $\{y(\tau + j\nu)\}_{j=-\infty}^{\infty} \in l^p(\mathbb{Z})$ for $\tau \in \mathscr{I} = [0, \nu)$. If y satisfies the uniform bound $||y(\tau + \cdot\nu)||_{l^p} < M$ for $\tau \in \mathscr{I}$, then $y \in L^p$.*

Proof.

$$\int_{-\infty}^{\infty} |y(s)|^p ds = \sum_{j=-\infty}^{\infty} \int_{j\nu}^{(j+1)\nu} |y(s)|^p ds = \sum_{j=-\infty}^{\infty} \int_0^{\nu} |y(\tau + j\nu)|^p d\tau. \tag{2.45}$$

By the uniform convergence of the sum, we interchange the sum and integral, and obtain the desired inequality:

$$\int_0^{\nu} \left(\sum_{j=-\infty}^{\infty} |y(\tau + j\nu)|^p \right) d\tau \leq \sup_{\tau \in \mathscr{I}} ||y(\tau + \cdot\nu)||_{l^p}^p < \infty. \tag{2.46}$$

□

Broadening the context of our discussion, we note the presence of analogous statements in the context of IVPs of neutral delay equations contained in Hale and Verduyn Lunel [23]. Let $D_0(r,B)$ refer to the difference operator D_0 with shifts $r = (r_1, ..., r_N)$ and coefficient matrices $B = (B_1, \ldots, B_n)$ in the form previously considered: $D_0(r,B)y_\xi = \sum_{j=1}^{N} B_j y_\xi(r_j)$. Within the purview of neutral delay equations, a constant coefficient difference operator D_0 is said to be **stable** if the eigenvalues λ are bounded away from the imaginary axis in the left-hand plane, i.e. $\mathbf{Re}\,\lambda \leq \bar{\lambda} < 0$. That is a stable difference operator D_0 is "stable" with regard to perturbations in the coefficient matrices B.

The operator $D_0(r,B)$ is said to be **strongly stable** if it is stable with respect to each perturbation $r \in (\mathbb{R}^+)^N$ in the delays. In this sense, a strongly stable operator is stable globally in the delays. The following theorem contained in [23] relates, among other relationships, a strongly stable difference operator D to the checkable condition (ii).

Theorem 2.2.9 (Theorem 6.1 from [23])**.** *The following statements are equivalent:*

(i) For some fixed $r \in (\mathbb{R}^+)^N$, $r = (r_1, r_2, \ldots, r_N)$ with $r_k > 0$ rationally independent, $D(r,B)$ is stable.

(ii) If $\gamma(A)$ is the spectral radius of a matrix A, then $\gamma_0(B) < 1$ where

$$\gamma_0(B) := \sup\{\gamma\left(\sum_{j=1}^{N} B_j e^{i\theta_j}\right) : \theta_j \in [0, 2\pi], j = 1, 2, \ldots N\}. \tag{2.47}$$

(iii) $D(r,B)$ is stable locally in the delays.

(iv) $D(r,B)$ is stable globally in the delays.

Switching back to the BVP context of neutral equations of mixed type we present the following definitions with analogy to stable and strongly stable operators in neutral delay equations:

Definition 2.2.10. *A difference operator D is **dichotomic** if D has an exponential dichotomy.*

Definition 2.2.11. *The difference operator D is **strongly dichotomic** if the coefficients B_j of D exhibit an exponential dichotomy over the hull $\mathscr{H}$*

$$\mathscr{H} := \{B_1 + \sum_{j=2}^{N} B_j e^{i\theta_j} | \theta_j \in [0, 2\pi], j = 2, \ldots, N\}.$$

When the shifts of difference operator are not rationally related (rationally independent), stronger assumptions need to be imposed. We believe the restrictive assumption of a strongly dichotomic operator makes the rationally independent case tractable.

2.3 Constant Coefficient Case

Theorem 2.3.1. *Consider the operator Λ_{D_0,L_0}, with D_0, L_0 given by (2.11). Assume that $\Delta_{D_0,L_0}(\lambda)$ is hyperbolic, hence $\Delta_{D_0,L_0}(i\eta)$ is invertible. In addition, assume that D_0 is dichotomic. Then Λ_{D_0,L_0} is an isomorphism from $W^{1,p}$ onto L^p for $1 \le p \le \infty$, with inverse given by convolution*

$$(\Lambda_{D_0,L_0}^{-1} h)(\xi) = (G_0 * h)(\xi) = \int_{-\infty}^{\infty} G_0(\xi - \eta) h(\eta) d\eta. \tag{2.48}$$

Moreover, the Green's function decays exponentially

$$|G_0(\xi)| \le K e^{-a_0|\xi|}. \tag{2.49}$$

Proof. Taking the Fourier transform of both sides of (2.12) we have

$$\begin{aligned}
(\widehat{\Lambda_{D_0,L_0}x})(\eta) &= \hat{h}(\eta)\\
\Delta_{D_0,L_0}(i\eta)\hat{x}(\eta) &= \hat{h}(\eta)\\
\hat{x}(\eta) &= \Delta^{-1}_{D_0,L_0}(i\eta)\hat{h}(\eta) \qquad (2.50)
\end{aligned}$$

noting that the hyperbolic characteristic matrix $\Delta_{D_0,L_0}(\lambda)$ is invertible along the imaginary axis.

Construct the function G_0, which will be the Green's function, by taking the inverse Fourier transform of $\Delta_{D_0,L_0}(i\eta)^{-1} \in L^2$ and giving the statement

$$G_0(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi\eta}\Delta_{D_0,L_0}(i\eta)^{-1}d\eta. \qquad (2.51)$$

Form Γ

$$\Gamma(\xi) = \sum_{j=1}^{N} B_{j,0}G_0'(\xi + r_j) - \sum_{j=1}^{N} A_{j,0}G_0(\xi + r_j). \qquad (2.52)$$

Now compute the Fourier Transform $\hat{\Gamma}$

$$\hat{\Gamma}(\eta) = [i\eta\sum_{j=1}^{N} B_{j,0}e^{i\eta r_j} - \sum_{j=1}^{N} A_{j,0}e^{i\eta r_j}]\hat{G}_0(\eta) = \Delta_{D_0,L_0}(i\eta)\hat{G}_0(\eta) = I \qquad (2.53)$$

That is, $\hat{\Gamma}(\eta) = \hat{\delta}I$, where δ denotes the delta distribution, and therefore we end up having, $\Gamma = \delta I$, which gives the formula

$$\sum_{j=1}^{N} B_{j,0}G_0'(\xi + r_j) = \sum_{j=1}^{N} A_{j,0}G_0(\xi + r_j) + \delta(\xi)I \qquad (2.54)$$

in the sense of tempered distributions. It follows that the following equation is satisfied almost everywhere

$$\sum_{j=1}^{N} B_{j,0} G_0'(\xi + r_j) = \sum_{j=1}^{N} A_{j,0} G_0(\xi + r_j) \qquad \text{a.e.} \tag{2.55}$$

From above we can also conclude that the function $F(\xi) = \sum_{j=1}^{N} B_{j,0} G_0(\xi + r_j)$ has a jump discontinuity at $\xi = 0$:

$$F(0+) - F(0-) = I, \text{ for } \xi = 0. \tag{2.56}$$

Since D_0 is dichotomic, there exists a vertical strip

$$\mathscr{S} = \{s | |\mathrm{Re}\ s| \leq a_0 < 1\} \tag{2.57}$$

for some a_0 about the imaginary axis in which both $B(s)$ and $\Delta_{D_0,L_0}(s)$ are invertible. Consider the following decomposition of $\Delta_{D_0,L_0}^{-1}(s)$ in order to accommodate the invertibility of $B(s)$

$$\Delta_{D_0,L_0}^{-1}(s) = ((s + a_0)B(s))^{-1} + R(s), \tag{2.58}$$

where $R(s) := \Delta_{D_0,L_0}^{-1}(s) - ((s+a_0)B(s))^{-1}$. For large $|\mathrm{Im}\ s|$, the boundedness of B, A, B^{-1} along the imaginary axis yield a Neumann series representation for $R(s)$:

$$R(s) = \frac{1}{s + a_0} \left(\sum_{k=1}^{\infty} (L(s))^k \right) B^{-1}(s), \tag{2.59}$$

for $L(s) = \frac{1}{s+a_0} B^{-1}(s)(a_0 B(s) + A(s))$ revealing that $|R(s)| = O(|\mathrm{Im}\ s|^{-2})$ uniformly in the strip $\mathscr{S}$. Since $B(s)$ and $\Delta_{D_0,L_0}(s)$ are invertible in the strip $\mathscr{S}$ it follows that $R(s)$ is holomorphic within the strip. By taking the inverse Fourier transform of $\Delta^{-1}_{D_0,L_0}(i\eta)$ we have the decomposition of G_0 into the sum of a conditionally convergent integral and absolutely convergent integral:

$$G_0(\xi) = E(\xi) + \mathscr{F}^{-1}[R(i\cdot)], \tag{2.60}$$

where $E(\xi)$ is a conditionally convergent inverse Fourier integral:

$$E(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\eta\xi}}{i\eta + a_0} B_0^{-1}(i\eta)d\eta. \tag{2.61}$$

(Case: Rationally related shifts on Left-Hand Side.) The function $B_0^{-1}(s)$ being holomorphic in the strip $\mathscr{S}$ gives an absolutely convergent Fourier series

$$B_0^{-1}(s) = \sum_{k=-\infty}^{\infty} C_k e^{\frac{2\pi k}{p}s}, \qquad \text{for } s \in \mathscr{S}, \tag{2.62}$$

with

$$\sum_{k=-\infty}^{\infty} |C_k| e^{\frac{2\pi k}{p}a} < \infty, \qquad \text{for } a = \text{Re } s,\ s \in \mathscr{S}. \tag{2.63}$$

The integral for $E(\xi)$ can now be computed

$$\begin{aligned} E(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\eta\xi}}{i\eta + a_0} \sum_{k=-\infty}^{\infty} C_k e^{\frac{2\pi k}{p} i\eta} d\eta \\ &= \sum_{k=-\infty}^{\infty} \frac{C_k}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\eta(\xi - \frac{2\pi k}{p})}}{\eta - ia_0} d\eta \\ &= \sum_{k=-\infty}^{\infty} C_k E_0(\xi - \frac{2\pi k}{p}), \end{aligned} \tag{2.64}$$

where

$$E_0(\xi) = e^{-a_0\xi} H(\xi), \tag{2.65}$$

and $H(\xi)$ denotes the Heaviside function. Because of the convergence in the strip (2.63), the pointwise bound of $E(\xi)$ is exponential. For $\xi \to +\infty$

$$\begin{aligned} |E(\xi)| &\leq \sum_{k=-\infty}^{\infty} |C_k| E_0(\xi - \frac{2\pi k}{p}) \\ &= \sum_{k=-\infty}^{\infty} |C_k| e^{-a_0(\xi - \frac{2\pi k}{p})} H(\xi - \frac{2\pi k}{p}) \\ &\leq e^{-a_0\xi} \sum_{k=-\infty}^{\infty} |C_k| e^{\frac{2\pi k}{p} a_0} = C e^{-a_0\xi}. \end{aligned} \tag{2.66}$$

To examine the limit when $\xi \to -\infty$, we can decompose ξ as $\xi = \frac{2\pi k}{p} + \xi_k$, where $\xi_k > 0$, and obtain the following bound using the absolute convergence of the Fourier series (2.63) in $\mathscr{S}$

$$\begin{aligned} |E(\xi)| &\leq \sum_{\xi > \frac{2\pi k}{p}} |C_k| e^{-a_0(\xi - \frac{2\pi k}{p})} = e^{\frac{a_0}{2}\xi} \sum_{\xi > \frac{2\pi k}{p}} |C_k| e^{-\frac{3a_0}{2}\xi} e^{\frac{2\pi k}{p} a_0} \\ &= e^{\frac{a_0}{2}\xi} \sum_{\xi > \frac{2\pi k}{p}} |C_k| e^{\frac{2\pi k}{p}(\frac{-a_0}{2})} \left(e^{\frac{-3}{2} a_0 \xi_k} \right) \leq e^{\frac{a_0}{2}\xi} \sum_{\xi > \frac{2\pi k}{p}} |C_k| e^{\frac{2\pi k}{p}(\frac{-a_0}{2})} = C' e^{\frac{a_0}{2}\xi} \end{aligned}$$

Now, we handle the absolutely convergent integral $\mathscr{F}^{-1}[R(i\eta)]$. If $\xi > 0$, we choose the contour integral to lie in the left-half plane, consisting of a rectangular contour with vertical sides coincident with the lines $s = i\eta$ and $s = -a_0 + i\eta$. Since $R(s)$

is holomorphic in the strip $|\text{Re } s| \le a_0$, we observe that the contour integral is 0:

$$\int_{-iN}^{iN} e^{s\xi} R(s) ds + \int_{iN}^{-a_0+iN} e^{s\xi} R(s) ds + \int_{-a_0+iN}^{-a_0-iN} e^{s\xi} R(s) ds + \int_{-a_0-iN}^{-iN} e^{s\xi} R(s) ds = 0. \tag{2.67}$$

Since we have the uniform decay estimate $|R(s)| = O(|\text{Im } s|^{-2})$ in the strip $|\text{Re } s| \le a_0$, we find that second integral goes to 0 as $N \to \infty$

$$\int_{iN}^{-a_0+iN} |e^{s\xi} R(s)| ds \le \int_{iN}^{-a_0+iN} |e^{s\xi}| \frac{C}{N^2} ds \le \frac{e^{-a\xi} - 1}{\xi} \frac{C}{N^2} \to 0 \text{ as } N \to \infty. \tag{2.68}$$

Similarly, the fourth integral goes to zero. The first and third integrals are equal, and by taking the limit $N \to \infty$ we find:

$$\int_{-\infty}^{\infty} e^{i\eta\xi} R(i\eta) d\eta = e^{-a_0\xi} \int_{-\infty}^{\infty} e^{i\eta\xi} R(-a_0 + i\eta) d\eta, \tag{2.69}$$

by making the change of variables $s = i\eta$ and $s = -a_0 + i\eta$, respectively. Similarly, if $\xi < 0$, the contour can be chosen in the right-half plane to exhibit exponential decay.

We observe that G_0 is the sum $G_0 = E(\xi) + \mathscr{F}^{-1}[R(i\eta)]$ of exponentially decaying functions in both cases of $\xi > 0$ and $\xi < 0$ putting $G_0 \in L^1$. Let $x = G_0 * h$. If $h \in L^p$ for $1 \le p \le \infty$, by Young's inequality we have $x \in L^p$.

To show that $x' \in L^p$ we first demonstrate that $G_0' \in L^1$. From (2.55), we observe that $\sum_j B_{j,0} G_0'(\xi + r_j) \in L^1$, since $G_0 \in L^1$. For each half line, the monotonically decreasing upper bound on G_0 implies that for each $\tau \in [0, \nu)$, the sampling at ν spaced intervals is in fact in $l^1(\mathbb{Z}, \mathbb{R}^d)$, where the j^{th} entry of f_k is given by $(f_k)_j = G_0(\tau + (j+k)\nu)$. It follows from Proposition 2.2.7, that $u_k = \{G'(\tau + (\cdot + k)\nu) \in l^1$. It follows from Lemma (2.2.8) that $G_0' \in L^1$, since $\|G_0'(\tau + \cdot \nu)\|_{l^1}$ is bounded for $\tau \in [0, \nu)$.

It is enough to show x' is equivalent to an L^p function in the sense of distributions to conclude that $x' \in L^p$. Utilizing distributions and Fubini's theorem we now show that x' is equivalent to a L^p function:

$$\begin{aligned}\int \zeta(\xi)x'(\xi)d\xi &= -\int \zeta'(\xi)x(\xi)d\xi \\ &= -\int \zeta'(\xi)\int G_0(\xi-\eta)h(\eta)d\eta d\xi \\ &= -\int\int \zeta'(\xi)G_0(\xi-\eta)d\xi h(\eta)d\eta.\end{aligned} \tag{2.70}$$

Note that the multiple jumps inherent to G_0 must be taken into account when integrating by parts. We denote the jump at $j\nu$ by $J(j\nu) = G_0(j\nu^+) - G_0(j\nu^-)$, and it follows that J must satisfy a consistency condition in (2.56):

$$\begin{aligned}I &= F(0^+) - F(0^-) \\ &= \sum_{i=1}^{N} B_{i,0}\Big(G_0(0^+ + r_i) - G_0(0^- + r_i)\Big) \\ &= \sum_{i=1}^{N} B_{i,0}J(r_i)\end{aligned} \tag{2.71}$$

Integrating by parts, we find

$$\int \zeta'(\xi)G_0(\xi-\eta)d\xi = -\sum_{j=-\infty}^{\infty}\zeta(\eta+j\nu)J(j\nu) - \int \zeta(\xi)G_0'(\xi-\eta)d\xi. \tag{2.72}$$

We already have $G_0' \in L^1$ implying $G' * h \in L^p$. Turning our attention to the summation, we make the change of variables $\mu = \eta + j\nu$, and we find

$$\int \sum_{j=-\infty}^{\infty} \zeta(\eta + j\nu) J(j\nu) h(\eta) d\eta = \int \sum_{j=-\infty}^{\infty} \zeta(\mu) J(j\nu) h(\mu - j\nu) d\mu$$
$$= \int \zeta(\mu) \sum_{j=-\infty}^{\infty} J(j\nu) h(\mu - j\nu) d\mu \tag{2.73}$$

All that remains to show $x' \in L^p$ is that the contribution from the sum is in L^p. From the exponential decay bounds on G_0, it follows that

$$|J(j\nu)| \leq Ce^{-a_0 |j| \nu}, \tag{2.74}$$

where C is some constant and a_0 is the same decay constant as the bound on G_0. From the triangle inequality and (2.74), we have the following

$$\left\| \sum_{j=-\infty}^{\infty} J(j\nu) h(\mu - j\nu) \right\|_{L^p} \leq C \sum_{j=-\infty}^{\infty} e^{-a_0 |j| \nu} \| h(\mu - j\nu) \|_{L^p} \tag{2.75}$$
$$= C \| h \|_{L^p} \sum_{j=-\infty}^{\infty} e^{-a_0 |j| \nu} < \infty, \tag{2.76}$$

where the sum is a convergent geometric series proving $x' \in L^p$.

To conclude the proof, we now show Λ_{D_0, L_0} is an isomorphism. For each $h(\xi) \in L^p$ it can be shown that $x = G * h$ satisfies the inhomogeneous equation (2.12) in terms of distributions. This implies that (2.12) is satisfied almost everywhere. In addition, the solution is also unique, since on the Fourier side the only solution to $\hat{\Delta}_{D_0, L_0} \hat{x} = 0$ is the zero solution, since $\hat{\Delta}_{D_0, L_0}$ is invertible along the imaginary axis. This proves Λ_{D_0, L_0} is an isomorphism concluding the proof.

□

The term $E(\xi)$ contained within the Green's function $G_0(\xi)$ exhibits many jump discontinuities. In the following cases p represents the periodicity of $B_0(i\eta)$, in the strip of invertibility $\mathscr{S} = \{s \big| |\text{Re } s| < \hat{a}_0\}$ $\hat{a}_0$ is the least upper bound on the distance from the imaginary axis, and a_0 is the chosen value of decay consistent with (2.58) and the subsequent discussion. The following four cases and corresponding plots demonstrate the discontinuous nature of the Green's function.

We consider the following left-hand sides:

1. This case exhibits diagonal dominance with rationally related shifts. The roots of $B_0(s)$ are $s = \pm 1 + i(\pi + 2\pi k)$, for $k \in \mathbb{Z}$

$$
\begin{aligned}
B_1 = 1, \quad B_2 = \varepsilon, \quad B_3 = \varepsilon, \\
r_1 = 0, \quad r_2 = +1, \quad r_3 = -1, \\
a_0 = .99, \quad \hat{a}_0 = 1, \quad p = 2\pi, \quad \text{and} \\
B_0(s) = 1 + 2\varepsilon \cosh(s), \quad \varepsilon = \frac{1}{2\cosh(1)}.
\end{aligned}
\tag{2.77}
$$

2. This case exhibits the function B_0 with a large periodicity due to a shift $r_2 = .7$ close to the irrational shift $1/\sqrt{2}$, i.e. within $|.7 - 1/\sqrt{2}| < .0072$

$$
\begin{aligned}
B_1 = 1, \quad B_2 = \varepsilon, \quad B_3 = \varepsilon, \\
r_1 = 0, \quad r_2 = +.7, \quad r_3 = -1, \\
a_0 = 1, \quad \hat{a}_0 \approx 1.05625, \quad p = 20\pi, \quad \text{and} \\
B_0(s) = 1 + \varepsilon(e^{.7s} + e^{-s}).
\end{aligned}
\tag{2.78}
$$

3. This case has a function B_0 with a shift that is not rationally related. It is well approximated by the previous case

$$\begin{aligned} B_1 &= 1, \qquad B_2 = \varepsilon, \quad B_3 = \varepsilon, \\ r_1 &= 0, \quad r_2 = +1/\sqrt{2}, \quad r_3 = -1, \\ a_0 &= 1, \quad \hat{a}_0 \approx 1.05308, \quad p = +\infty, \quad \text{and} \\ B_0(s) &= 1 + \varepsilon(e^{\frac{s}{\sqrt{2}}} + e^{-s}). \end{aligned} \tag{2.79}$$

4. This case exhibits diagonal dominance by a factor of $\varepsilon_2 = .1$ giving rise to roots $s = \pm.314925 \pm 2\pi i$, and $E(\xi)$ decays slowly

$$\begin{aligned} B_1 &= 2 + \varepsilon_2, \qquad B_2 = -1, \quad B_3 = -1, \\ r_1 &= 0, \qquad r_2 = +1, \quad r_3 = -1, \\ a_0 &= .25, \quad \hat{a}_0 \approx .314925, \quad p = 2\pi, \quad \text{and} \\ B_0(s) &= \varepsilon + 2(1 - \cosh(s)), \quad \varepsilon_2 = .1. \end{aligned} \tag{2.80}$$

In the plots of $E(\xi)$, depicted in Figure 2.1, Heaviside functions are present every $s = 2\pi/p$ units of varying magnitudes in the periodic cases. Case 3, which shifts are not rationally related, appears to be a small perturbation away from the nearby rationally related Case 2.

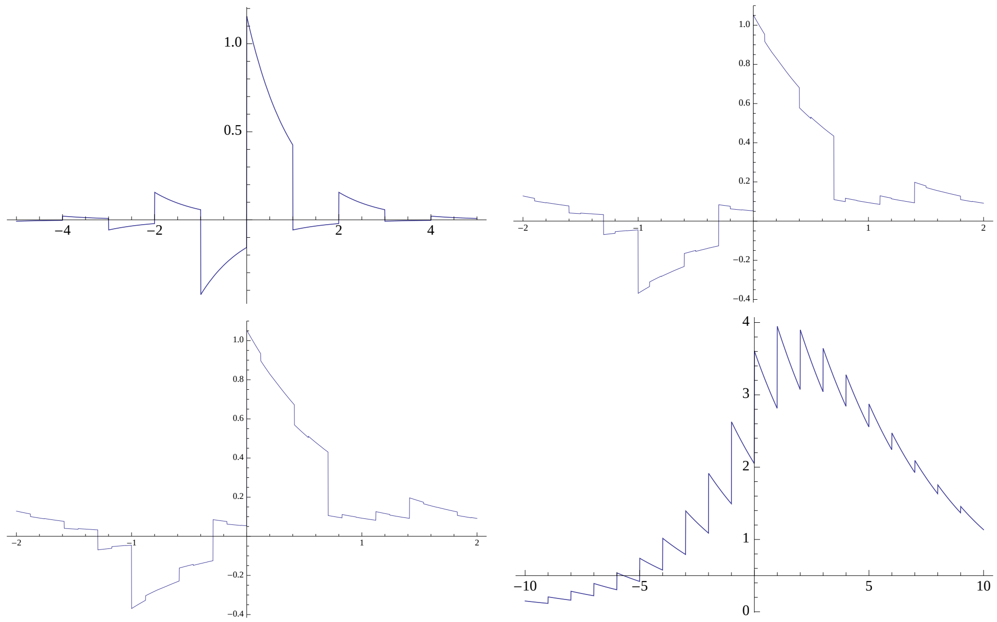


Figure 2.1: Examples of the discontinuous nature of $E(\xi)$. The upper left plot corresponds to Case 1, the upper right plot corresponds to Case 2, and so on.

2.4 Variable Coefficient Case

In this section the coefficients may depend on the independent variable ξ. Specifically, we are trying to solve $\Lambda_{D,L}x = h$:

$$\sum_{j=1}^{N} B_j(\xi)x'(\xi + r_j) = \sum_{j=1}^{N} A_j(\xi)x(\xi + r_j) + h(\xi). \tag{2.81}$$

The equation $\Lambda_{D,L}x = h$, with $\Lambda_{D,L}x = (\Lambda_{D_0,L_0} + M)x$ can be rewritten using the decomposition M into M_1 and M_2, with

$$\begin{aligned} Mx &= M_1x - M_2x & (2.82)\\ &= \sum_{j=1}^{N} M_{1,j}(\xi)x'(\xi + r_j) - \sum_{j=1}^{N} M_{2,j}(\xi)x(\xi + r_j), & (2.83)\end{aligned}$$

and noting that each matrix coefficient $M_{i,j}$ is uniformly small: $||M_{i,j}(\xi)|| < \varepsilon$.

To find the solution to the variable coefficient problem we will utilize a Neumann series expansion of $(I+\Lambda_{D_0,L_0}^{-1}M)^{-1}$ provided that $||\Lambda_{D_0,L_0}^{-1}M|| < 1$.

Proposition 2.4.1. *If the constant coefficient operator pair (D_0, L_0) is hyperbolic with dichotomic D_0, then there exists constants ε, K and a such that if the operator M is uniformly small*

$$||M(\xi)|| \leq \varepsilon, \qquad \forall \xi \in \mathbb{R} \tag{2.84}$$

then $\Lambda_{D,L} : W^{1,p} \to L^p$ is an isomorphism for $1 \leq p \leq \infty$. In addition, there exists a Green's function $G : \mathbb{R}^2 \to \mathbb{C}^2$ such that

$$|G(\xi,\eta)| \leq K e^{-a|\xi-\eta|}, \tag{2.85}$$

where

$$(\Lambda_{D,L}^{-1}h)(\xi) = \int_{-\infty}^{\infty} G(\xi,\eta)d\eta \tag{2.86}$$

for any L^p.

Proof. By decomposing the operator $\Lambda_{D,L}$ into the sum of a constant coefficient operator plus a small perturbation we have $\Lambda_{D,L} = (\Lambda_{D_0,L_0} + M)$. Utilizing the invertibility of Λ_{L_0} gives the rewriting

$$h = \Lambda_{D,L}x = (\Lambda_{D_0,L_0} + M)x = \Lambda_{D_0,L_0}(I + \Lambda_{D_0,L_0}^{-1}M)x. \tag{2.87}$$

Provided that $||\Lambda_{D_0,L_0}^{-1}M|| < 1$ the expression $I+\Lambda_{D_0,L_0}^{-1}M$ is invertible and a Neumann series representation can be used to solve the following system

$$\begin{aligned} &\Lambda_{D_0,L_0}y = h, \\ &(I+\Lambda_{D_0,L_0}^{-1}M)x = y. \end{aligned} \tag{2.88}$$

To check that the norm $||\Lambda_{D_0,L_0}^{-1}M||$ is suitable small we identify the operator $Z: W^{1,p} \to W^{1,p}$ with

$$Z = \Lambda_{D_0,L_0}^{-1}M, \quad \text{with} \quad ||Z|| = \sup_{||w||=1} ||\Lambda_{D_0,L_0}^{-1}Mw||. \tag{2.89}$$

We find that as long as

$$||M|| < \frac{1}{||\Lambda_{D_0,L_0}^{-1}||} =: \varepsilon, \tag{2.90}$$

then $||Z|| < 1$. Under this condition, Neumann series can be used, and then

$$x = (I+\Lambda_{D_0,L_0}^{-1}M)^{-1}\Lambda_{D_0,L_0}^{-1}h. \tag{2.91}$$

By expanding the Neumann series and bounding each term exponential bounds can be obtained. □

2.5 Results

2.5.1 Fredholm Alternative

Theorem 2.5.1 (Analogue of Theorem A from [31])**.** *Assume that the variable coefficient equation (2.4) is asymptotically hyperbolic with dichotomic D, then for each p* *with $1 \leq p \leq \infty$, the operator $\Lambda_{D,L}$ from $W^{1,p}$ to L^p is a Fredholm operator. The kernel of $\mathscr{K}_{D,L} \subseteq W^{1,p}$ of $\Lambda_{D,L}$ is independent of p, while the kernel $\mathscr{K}_{D^*,L^*}$ of the adjoint operator pair (2.15) is independent of q. The range $\mathscr{R}^p_{D,L} \subseteq L^p$ of $\Lambda_{D,L}$ is given by*

$$\mathscr{R}^p_{D,L} = \left\{ h \in L^p \,\middle|\, \int_{-\infty}^{\infty} \overline{y(\xi)} h(\xi) d\xi = 0, \text{ for all } y \in \mathscr{K}_{D^*,L^*} \right\}. \tag{2.92}$$

In particular,

$$\begin{aligned} \dim \mathscr{K}_{D^*,L^*} &= \operatorname{codim} \mathscr{R}^p_{D,L}, \\ \dim \mathscr{K}_{D,L} &= \operatorname{codim} \mathscr{R}^p_{D^*,L^*}, \\ \operatorname{ind}(\Lambda_{D,L}) &= -\operatorname{ind}(\Lambda_{D^*,L^*}). \end{aligned} \tag{2.93}$$

If $\Lambda_{D,L}$ is a hyperbolic constant coefficient operator Λ_{D_0,L_0}, then

$$\begin{aligned} \operatorname{codim} \mathscr{R}^p_{D_0,L_0} &= 0, \\ \dim \mathscr{K}_{D_0,L_0} &= 0, \\ \operatorname{ind}(\Lambda_{D_0,L_0}) &= 0, \end{aligned} \tag{2.94}$$

implying that Λ_{D_0,L_0} is an isomorphism.

Proof. To show that $\Lambda_{D,L}$ is a Fredholm operator three things must be shown: (i) the kernel $\mathscr{K}_{D,L}$ is finite dimensional, (ii) the range $\mathscr{R}_{D,L}$ is closed, and (iii) $\mathscr{R}_{D,L}$ has finite codimension in L^p.

By working directly with $\Lambda_{D,L}$ instead of the just the operator L the arguments in [31] can be modified, and the same general framework applies. The facts that (i) the kernel $\mathscr{K}_{D,L}$ is finite dimensional and that (ii) the range $\mathscr{R}_{D,L}$ is closed follow from Lemma 5.4 and Corollaries 5.5, 5.6 in [31].

To show that (iii) the range $\mathscr{R}^p_{D,L}$ has finite codimension in L^p, the range can be shown to be equal to the set $\mathscr{Q}^p_{D,L}$ where

$$\mathscr{Q}^p_{D,L} = \left\{ h \in L^p \,\middle|\, \int_{-\infty}^{\infty} \overline{y(\xi)} h(\xi) d\xi = 0, \text{ for all } y \in \mathscr{K}_{D^*,L^*} \right\}. \tag{2.95}$$

By modifying the form of the adjoint operator for the neutral case, the argument follows from [31]. The inclusion $\mathscr{R}^p_{D,L} \subseteq \mathscr{Q}^p_{D,L}$ is a consequence of the definition of the adjoint operator. In the case $1 \leq p < \infty$, the other inclusion $\mathscr{R}^p_{D,L} \supseteq \mathscr{Q}^p_{D,L}$ follows from a contradiction argument. The element $y \in L^q$ of the dual space to $\mathscr{R}^p_{D,L} \subseteq L^p$ is assumed to map all elements $\Lambda_{D,L}x$ to zero, but there exists $h \in \mathscr{Q}^p_{D,L}$ such that $\int \overline{y(\xi)} h(\xi) d\xi \neq 0$. The contradiction follows when $y \in \mathscr{K}_{D^*,L^*}$. The $p = \infty$ case also follows from the dichotomic assumption on D. $\square$

2.5.2 Cocycle Property

Theorem 2.5.2 (Analogue of Theorem B from [31])**.** *Assume the variable coefficient equation (2.4) is asymptotically hyperbolic, with the associated left-hand side difference operator being dichotomic. Then, the Fredholm index of $\Lambda_{D,L}$ depends only on the limiting operator pairs $(D_\pm, L_\pm)$ as it is independent of the path between, given from $D_\pm = \lim_{\xi \to \pm\infty} D(\xi)$ (resp. $L_\pm$).*

Denote the Fredholm index by

$$\operatorname{ind}(\Lambda_{D,L}) = i(\Lambda_{D_-,L_-}, \Lambda_{D_+,L_+}), \tag{2.96}$$

where Λ_{D_-,L_-} is the autonomous operator with the limiting constant coefficient operator pair (D_-, L_-). For any three constant coefficient hyperbolic operator pairs

$P_1 = (D_1, L_1), P_2 = (D_2, L_2)$, and $P_3 = (D_3, L_3)$ we have

$$i(P_1, P_2) + i(P_2, P_3) = i(P_1, P_3). \tag{2.97}$$

Proof. To show that the Fredholm operator $\Lambda_{D,L}$ only depends on the limiting operators $(D_\pm, L_\pm)$ consider two asymptotically hyperbolic systems with the operator pairs $(D^\rho(\xi), L^\rho(\xi))$ for $\rho = 0, 1$

$$\begin{aligned} D^0(\xi)x'_\xi &= L^0(\xi)x_\xi, \\ D^1(\xi)x'_\xi &= L^1(\xi)x_\xi, \end{aligned} \tag{2.98}$$

with the same limiting operators pair $(D_+, L_+) = (D^0_+, L^0_+) = (D^1_+, L^1_+)$, and similarly for '-'. Further, assume that D^ρ, $\rho = 0, 1$ is dichotomic. The linear homotopy $\Lambda_{D^\rho, L^\rho} x_\xi = D^\rho(\xi)x'_\xi - L^\rho(\xi)x_\xi = 0$ for $0 \leq \rho \leq 1$ links the two systems in (2.98), where upon expansion gives

$$((1-\rho)D^0(\xi) + \rho D^1(\xi))x' = ((1-\rho)L^0(\xi) + \rho L^1(\xi))x. \tag{2.99}$$

The homotopy is asymptotically hyperbolic for each $0 \leq \rho \leq 1$. The operator Λ_{D^ρ, L^ρ} is Fredholm for each $\rho \in [0, 1]$ if D^ρ can be shown to be dichotomic. Since $D_\pm$ are the limiting values for both D^0 and D^1, the same projections and $\tilde{\lambda}_1, \tilde{\lambda}_2$ values are valid. Hence, D^ρ is dichotomic with the same values $\tilde{\lambda}_1, \tilde{\lambda}_2$.

The homotopy Λ_{D^ρ, L^ρ} varies continuously with ρ in $\mathscr{L}(W^{1,p}, L^p)$, and since the Fredholm index is constant on connected regions, it follows that the index is homotopy invariant due to its continuity. The index does, in fact, only depend on the limiting operators: $\operatorname{ind}(\Lambda_{D^0, L^0}) = \operatorname{ind}(\Lambda_{D^1, L^1})$.

For the cocycle property consider three hyperbolic pairs P_1, P_2, and P_3

$$(1-H(\xi))\begin{pmatrix}\Lambda_{P_1} & 0\\ 0 & \Lambda_{P_2}\end{pmatrix}\begin{pmatrix}x(\xi)\\ y(\xi)\end{pmatrix}+H(\xi)R(\rho)\begin{pmatrix}\Lambda_{P_2} & 0\\ 0 & \Lambda_{P_3}\end{pmatrix}R(-\rho)\begin{pmatrix}x(\xi)\\ y(\xi)\end{pmatrix}=0, \tag{2.100}$$

where $H(\xi)$ is a the Heaviside function, and utilizing the rotation matrix $R(\rho)$ in [31] given by

$$R(\rho)=\begin{pmatrix}\cos(\frac{\pi\rho}{2})I_d & \sin(\frac{\pi\rho}{2})I_d\\ -\sin(\frac{\pi\rho}{2})I_d & \cos(\frac{\pi\rho}{2})I_d\end{pmatrix}.$$

For any $\rho \in [0,1]$, the characteristic equations for the limiting operators are given by

$$\begin{aligned}\xi=-\infty,\quad & \det\begin{pmatrix}\Delta_{P_1}(\lambda) & 0\\ 0 & \Delta_{P_2}(\lambda)\end{pmatrix}=\det\Delta_{P_1}(\lambda)\det\Delta_{P_2}(\lambda)=0,\\ \xi=+\infty,\quad & \det\left(R(\rho)\begin{pmatrix}\Delta_{P_2}(\lambda) & 0\\ 0 & \Delta_{P_3}(\lambda)\end{pmatrix}R(-\rho)\right)=\det\Delta_{P_2}(\lambda)\det\Delta_{P_3}(\lambda)=0,\end{aligned} \tag{2.101}$$

since $R(\rho)$ is orthogonal. In addition, each operator Λ_{P_j} is hyperbolic, which implies that the system of equations Λ_{D^ρ,L^ρ} is asymptotically hyperbolic for each ρ.

Further consideration is needed to verify that a dichotomy is maintained along the homotopy path, and yield the fact that Λ_{D^ρ,L^ρ} is ultimately Fredholm for each $\rho \in [0,1]$. By replacing $\Delta_{P_j}(\lambda)$ with $B_j(\lambda)$ for the associated difference operator D_j for $j=1,2,3$ in (2.101), one can see that the dichotomy is maintained.

Λ_{D^ρ,L^ρ} varies continuously, and is hence independent of ρ. When the system decouples, the Fredholm index can be calculated as the sum of indices of the x and y equations. The following behavior of the homotopy is exhibited at the limiting values of $\xi = \pm\infty$:

$$
\begin{aligned}
&\xi = -\infty, \quad \begin{pmatrix} \Lambda_{P_1} x \\ \Lambda_{P_2} y \end{pmatrix} = 0, && 0 \le \rho \le 1, \\
&\xi = +\infty, \quad \begin{pmatrix} \Lambda_{P_2} x \\ \Lambda_{P_3} y \end{pmatrix} = 0, \quad \begin{pmatrix} \Lambda_{P_3} x \\ \Lambda_{P_2} y \end{pmatrix} = 0, && \rho = 0, 1 \text{ resp.}
\end{aligned}
\tag{2.102}
$$

By summing the indices associated with the x and y equations we find that

$$
\begin{aligned}
\operatorname{ind}(\Lambda_{D^0,L^0}) &= i(P_1,P_2) + i(P_2,P_3) \\
\operatorname{ind}(\Lambda_{D^1,L^1}) &= i(P_1,P_3) + i(P_2,P_2) = i(P_1,P_3),
\end{aligned}
\tag{2.103}
$$

giving the cocycle property.

□

2.5.3 Spectral Flow Property

In order to relate the Fredholm index to the number of eigenvalues crossing the imaginary axis, we present the analogous machinery for the neutral case adapted from [31]. For the N-tuples of constant coefficient matrices

$$
\begin{aligned}
\mathbf{A} &= (A_{1,0}, A_{2,0}, \ldots, A_{N,0}) \in (\mathbb{C}^{d\times d})^N \\
\mathbf{B} &= (B_{1,0}, B_{2,0}, \ldots, B_{N,0}) \in (\mathbb{C}^{d\times d})^N
\end{aligned}
\tag{2.104}
$$

let σ_B, σ_A be homotopies varying among these constant coefficient matrices

$$\sigma_A, \sigma_B : [-1,1] \to (\mathbb{C}^{d\times d})^N, \tag{2.105}$$

where $\sigma_A(\pm 1) = \mathbf{A}_\pm$ and $\sigma_B(\pm 1) = \mathbf{B}_\pm$. For a continuous path $\sigma(\rho) = (\sigma_B(\rho), \sigma_A(\rho))$ define the set of nonhyperbolic points in ρ to be

$$\begin{aligned} NH(\sigma) = \{\rho \in [-1,1] |\ &\text{Eq. (2.13) with coefficients } (\mathbf{B},\mathbf{A}) = (\sigma_B(\rho), \sigma_A(\rho)) \\ &\text{where } \mathbf{B} \text{ is dichotomic, but the pair } (\mathbf{B},\mathbf{A}) \text{ is not hyperbolic}\}. \end{aligned} \tag{2.106}$$

The computation of the number of eigenvalues crossing the imaginary axis relies upon arriving at a homotopy from $(\mathbf{B}_-, \mathbf{A}_-)$ to $(\mathbf{B}_+, \mathbf{A}_+)$ in which (1) the eigenvalues cross one at a time, and (2) the eigenvalues cross transversely. By requiring σ_B to maintain a dichotomy along the homotopy path, the situation where a countably infinite number of eigenvalues crossing the imaginary axis is precluded. With these properties in mind, we present the analogous definitions to those in [31]:

Definition 2.5.3. *A constant coefficient system (2.13) satisfies* ***Property*** $\mathscr{G}_0$ *if there exists at most one root* $\lambda = i\eta$ *on the imaginary axis of the characteristic equation* $\det \Delta_{D_0,L_0}(\lambda) = 0$.

Definition 2.5.4. *Let* $\sigma = (\sigma_B, \sigma_A) \in C^1([-1,1], (\mathbb{C}^{d\times d})^N \times (\mathbb{C}^{d\times d})^N)$ *be a smooth one-parameter family of coefficients for (2.13). The family of equations satisfies* ***Property*** $\mathscr{G}_1$ *if*

1. *for each* $\rho \in [-1,1]$ *eq. (2.13) with coefficients of dichotomic* $\boldsymbol{B} = \sigma_B(\rho)$ *and* $\boldsymbol{A} = \sigma_A(\rho)$ *satisfies Property* $\mathscr{G}_0$*,*

2. *for* $\rho = \pm 1$ *eq. (2.13) is hyperbolic, and*

3. *all eigenvalues $\lambda = \lambda(\rho)$ on the imaginary axis Re $\lambda(\rho_0) = 0$ for some $\rho_0 \in (-1,1)$ cross the imaginary axis transversely, i.e. Re $\lambda'(\rho_0) \neq 0$.*

Also, let $S_\gamma : (\mathbb{C}^{d\times d})^N \to (\mathbb{C}^{d\times d})^N$ be the map defined for each $\gamma \in \mathbb{R}$

$$S_\gamma(B_1,B_2,\dots,B_N,A_1,A_2,\dots,A_N) = \\ (B_1,e^{-\gamma r_2}B_2,\dots,e^{-\gamma r_N}B_N,A_1+\gamma B_1,e^{-\gamma r_2}(A_2+\gamma B_1),\dots,e^{-\gamma r_N}(A_N+\gamma B_N)). \tag{2.107}$$

This transformation is a result of the change of variables $y(\xi) = e^{\gamma\xi}x(\xi)$, and as in [31], the map S_γ is consistent with a rigid shift of the spectrum

$$\Delta_{S_\gamma(\mathbf{B},\mathbf{A})}(\lambda) = \Delta_{(\mathbf{B},\mathbf{A})}(\lambda-\gamma). \tag{2.108}$$

Theorem 2.5.5 ((Spectral Flow Property) Analogue of Theorem C from [31])**.** *Let (D^ρ,L^ρ), for $-1 \leq \rho \leq 1$ be a continuously varying one-parameter family of constant coefficient operators with D^ρ dichotomic for $-1 \leq \rho \leq 1$, and suppose that the operator pairs $(D_\pm,L_\pm)$ are hyperbolic. Suppose further that there are finitely many values*

$$\{\rho_1,\rho_2,\dots,\rho_J\} \subseteq (-1,1) \tag{2.109}$$

of ρ for which (D^ρ,L^ρ) is not hyperbolic. Then

$$i(P_-,P_+) = -cross(P_\rho) \tag{2.110}$$

is the net number of eigenvalues which cross the imaginary axis from left to right as ρ increases from -1 to 1.

Proof. The same general framework from [31] applies to the neutral case. By requiring D^ρ to be dichotomic, pathological behavior is excluded, and by taking into consideration the **B** coefficient matrices the same theory applies.

The general framework of the proof is as follows: first, let $\sigma(\rho)$ denote the family of coefficients. By Proposition 8.1 in [31] the homotopy can be assumed to satisfy the property $\mathscr{G}_1$, by a density argument. From Proposition 8.3 in [31] the homotopy σ can be identified with another homotopy $\tilde{\sigma}$ with the same (i) limiting values $\tilde{\sigma}(\pm 1) = \sigma(\pm 1)$, (ii) parameter values, ρ_j, of non-hyperbolicity, and (iii) eigenvalue crossing rates, i.e. $\mu_j := \text{Re } \lambda'(\rho_j) =: \tilde{\mu_j}$. Moreover, the utility of the homotopy $\tilde{\sigma}$ stems from a rigid shift in the spectrum across the imaginary axis, where $\tilde{\sigma}$ takes the form $\tilde{\sigma}(\rho) = S_{\mu_j(\rho-\rho_j)}\tilde{\sigma}(\rho_j)$.

The homotopy $\tilde{\sigma}$ between the limiting operators Λ_{D_-,L_-} and Λ_{D_+,L_+} can, for small ε, be decomposed in the same manner as the explicit case with the interpretation that $P_\pm$ is shorthand for the operator pair $(D_\pm, L_\pm)$

$$\begin{aligned} &i(P_-,P_+) = \\ &i(P_-,\tilde{P}^{\rho_1-\varepsilon}) + \sum_{j=1}^{J-1} i(\tilde{P}^{\rho_j+\varepsilon},\tilde{P}^{\rho_{j+1}-\varepsilon}) + \sum_{j=1}^{J} i(\tilde{P}^{\rho_j-\varepsilon},\tilde{P}^{\rho_j+\varepsilon}) + i(\tilde{P}^{\rho_J+\varepsilon},P_+). \end{aligned} \tag{2.111}$$

The rest of argument follows as in [31]. The cocycle property is applied to show that the terms $i(P_-,\tilde{P}^{\rho_1-\varepsilon})$, $i(\tilde{P}^{\rho_J+\varepsilon},P_+)$ and each term in the sum $\sum_{j=1}^{J-1} i(\tilde{P}^{\rho_j+\varepsilon},\tilde{P}^{\rho_{j+1}-\varepsilon})$ are zero since no eigenvalues cross the imaginary axis. The index reduces to

$$i(P_-,P_+) = \sum_{j=1}^{J} i(\tilde{P}^{\rho_j-\varepsilon},\tilde{P}^{\rho_j+\varepsilon}) = \sum_{j=1}^{J} i(S_{-\mu_j\varepsilon}\tilde{P}^{\rho_j}, S_{\mu_j\varepsilon}\tilde{P}^{\rho_j+\varepsilon}) \tag{2.112}$$

where the terms in the last sum can be rewritten using Proposition 8.4 in [31]. Proposition 8.4 states that the index of a rigid shift $i(S_{-\gamma}P, S_\gamma P) = -\text{sgn } \gamma$. In conclusion, it is found that $\sum_{j=1}^{J} i(S_{-\mu_j \varepsilon}\tilde{P}^{\rho_j}, S_{\mu_j \varepsilon}\tilde{P}^{\rho_j+\varepsilon}) = -\sum \operatorname{sgn} \mu_j$, which has the interpretation as the net number of crossing eigenvalues.

□

2.6 Outline of Examples

To illustrate the Fredholm theory developed in the previous sections we will outline the details involved with applying the theory.

2.6.1 Models Considered

The following nonlinear mixed type traveling wave equation will be considered

$$-c\sum_{j=1}^{N} B_j(\alpha)\phi'(\xi+r_j) = \sum_{j=1}^{N} A_j(\alpha)\phi(\xi+r_j) - \sum_{j=1}^{N} C_j(\alpha) f(\phi(\xi+r_j)), \tag{2.113}$$

where a traveling wave solution is a pair (ϕ_α, c_α) of traveling wave profile(s) and a scalar wave speed both depending on the parameter α. Two particular cases are of interest:

1. $C_j(\alpha) = B_j(\alpha)$ for $1 \leq j \leq N$, where the equation takes the following form

$$\sum_{j=1}^{N} B_j(\alpha)[-c\phi'(\xi+r_j) + f(\phi(\xi+r_j))] = \sum_{j=1}^{N} A_j(\alpha)\phi(\xi+r_j); \tag{2.114}$$

2. $C_1(\alpha) = I$ with $C_j = 0$ for $j \neq 1$, where the equation takes the form

$$-c\sum_{j=1}^{N} B_j(\alpha)\phi'(\xi+r_j) = \sum_{j=1}^{N} A_j(\alpha)\phi(\xi+r_j) - f(\phi(\xi)). \tag{2.115}$$

The first case arises in the context of modeling current flow in an infinite chain of cells modeling a nerve fiber. This case is considered in detail in Chapter 3, and in particular see equation (3.6). The second case has only one particular instance of the nonlinearity in the equation, and may be of a more familiar type.

The rest of this section discusses the steps pertaining to local continuation, given a known solution (ϕ_0, c_0) for $\alpha = \alpha_0$ to the nonlinear problem. For example, in Chapter 3 we examine a problem with $N = 3$, $r_1 = 0, r_2 = -1, r_3 = +1$, and in general we have in mind a block tridiagonal or banded structure for the infinite dimensional matrix B as described in the context of (2.3).

The following boundary conditions

$$\lim_{\xi \to -\infty} \phi(\xi) = 0, \text{ and } \lim_{\xi \to \infty} \phi(\xi) = 1, \tag{2.116}$$

are imposed for the problems considered, and upon linearization we utilize the notation $\beta(\xi) = f'(\phi_0(\xi))$. The asymptotic limits of β are denoted by $\beta_\pm := \lim_{\xi \to \pm\infty} \beta(\xi)$. The cubic and the rational, cubic-like [1] bistable nonlinearities are considered in sections 2.7 and 3-3.1, respectively:

Cubic

$$f_c(u) = u(u-a)(u-1), \tag{2.117}$$

Rational, Cubic-Like

$$f_{cl}(u) = \left(\frac{d\gamma(2u-1)}{\gamma[1-u]u+1} - cb \right) u(u-1), \text{ with} \tag{2.118}$$

$$\begin{aligned} b &= \tfrac{1+3(2a-1)}{\sqrt{2d}}, \qquad \gamma = 2(\cosh(b)-1), \\ c &= \tfrac{d(2a-1)}{b} \tfrac{\gamma}{\gamma[1-a]a+1}. \end{aligned} \tag{2.119}$$

For the cubic nonlinearity the asymptotic values of $\beta(\xi)$ are as follows

$$\begin{aligned} \beta_+ &= \lim_{\xi\to\infty} f_c'(\phi_0(\xi)) = f_c'(1) = 1-a, \\ \beta_- &= \lim_{\xi\to-\infty} f_c'(\phi_0(\xi)) = f_c'(0) = a. \end{aligned} \tag{2.120}$$

The detuning parameter a lies in the open interval $(0,1)$ giving the inequality $\beta_\pm > 0$, and hence bistability. The cubic-like nonlinearity can be viewed as

$$f(u) = h(u)u(u-1), \text{ where } h(u) = \left(\frac{d\gamma(2u-1)}{\gamma[1-u]u+1} - cb \right), \text{ and with} \tag{2.121}$$

$$\begin{aligned} f'(u) &= (2u-1)h(u) + u(u-1)h'(u), \\ f''(u) &= 2h(u) + 2(2u-1)h'(u) + u(u-1)h''(u). \end{aligned} \tag{2.122}$$

The following asymptotic values for $\beta_\pm$ can now be identified

$$\begin{aligned} \beta_+ &= \lim_{\xi\to\infty} f_{cl}'(\phi_0(\xi)) = f_{cl}'(1) = h(1) = d\gamma - cb, \\ \beta_- &= \lim_{\xi\to-\infty} f_{cl}'(\phi_0(\xi)) = f_{cl}'(0) = -h(0) = d\gamma + cb. \end{aligned} \tag{2.123}$$

We note that cb has the following bound

$$|cb| = d\gamma \left| \frac{2a-1}{\gamma[1-a]a+1} \right| < d\gamma, \text{ for } a \in (0,1),$$

which implies $\beta_\pm > 0$, and truly is a bistable nonlinearity.

The associated linearized equation about ϕ_0 for the values $c = c_0$ and $\alpha = \alpha_0$ is:

$$-c_0 \sum_{j=1}^{N} B_j(\alpha_0)\phi'(\xi + r_j) = \sum_{j=1}^{N} (A_j(\alpha_0) - C_j(\alpha_0)\beta(\xi + r_j))\phi(\xi + r_j). \tag{2.124}$$

As a shorthand we make the identification $\tilde{A}_j(\xi;\alpha) = A_j(\alpha) - C_j(\alpha)\beta(\xi + r_j)$ to collect terms, and we note that after linearizing, the matrix coefficients $\tilde{A}_j(\xi;\alpha)$ on the right hand side are no longer constant in ξ giving a linear non-autonomous equation. The linear operators $D(\xi)$ and $L(\xi)$ are defined in accordance with (2.8) and (2.9)

$$D(\xi)\phi'_\xi = \sum_{j=1}^{N} B_j(\alpha_0)\phi'(\xi + r_j), \quad L(\xi)\phi_\xi = \sum_{j=1}^{N} \tilde{A}_j(\xi;\alpha_0)\phi(\xi + r_j). \tag{2.125}$$

In the case that $D(\xi)$ does not depend on ξ we have $D(\xi) = D$. We continue introducing more notation by noting that associated to the homogeneous equation (2.124) is the linear operator $\Lambda_{c_0,D,L} : W^{1,p} \to L^p$

$$(\Lambda_{c_0,D,L}\phi)(\xi) = -c_0 \sum_{j=1}^{N} B_j(\alpha_0)\phi'(\xi + r_j) - \sum_{j=1}^{N} \tilde{A}_j(\xi;\alpha_0)\phi(\xi + r_j). \tag{2.126}$$

The operator $\Lambda_{c_0,D,L}$ is asymptotically autonomous, since $\beta(\xi) \to \beta_\pm$ for $\xi \to \pm\infty$, and the limiting operators at $\pm\infty$ are given by

$$(\Lambda_{c_0,D_\pm,L_\pm}\phi)(\xi) = -c_0 \sum_{j=1}^{N} B_{j,\pm}(\alpha_0)\phi'(\xi + r_j) - \sum_{j=1}^{N} \tilde{A}_{j,\pm}(\alpha_0)\phi(\xi + r_j). \tag{2.127}$$

We have $D_\pm = D$, since $B_{j,\pm} = B_j$ do not depend on ξ, and $\tilde{A}_{j,+}(\alpha_0) = A_j(\alpha_0) - C_j(\alpha_0)\beta_+$ with $\tilde{A}_{j,-}(\alpha_0) = A_j(\alpha_0) - C_j(\alpha_0)\beta_-$.

2.6.2 Characteristic Matrix

For each of the limiting operators in (2.127) a characteristic equation is obtained by letting Λ_{c_0,D_0,L_0} act on the form $\phi(\xi) = e^{\lambda\xi}$, where 0 refers to either $+$ or $-$, subsequently factoring the common term $e^{\lambda\xi}$ to obtain the characteristic matrix

$$\Delta_{D_0,L_0}(\lambda) = -c_0\lambda\left(\sum_{j=1}^{N} B_{j,0}(\alpha_0)e^{r_j\lambda}\right) - \sum_{j=1}^{N} \tilde{A}_{j,0}(\alpha_0)e^{r_j\lambda}, \tag{2.128}$$

and taking determinants to obtain the characteristic equation

$$\det(\Delta_{D_0,L_0}(\lambda)) = 0. \tag{2.129}$$

2.6.3 Asymptotically Hyperbolic

To verify that the operator (2.126) is asymptotically hyperbolic the limiting operators (2.127) both need to satisfy the condition $\det\Delta_{D_\pm,L_\pm}(i\eta) \neq 0$, $\forall\eta \in \mathbb{R}$. In particular, for scalar equations, substituting $\lambda = i\eta$ into the characteristic equation and showing that the imaginary and real parts of the characteristic equation are never simultaneously zero gives the desired result.

2.6.4 Diagonal Dominance and Dichotomy

In addition to asymptotic hyperbolicity, to meet the conditions of the Fredholm Alternative theorem the associated matrix-valued function $B(s)$ also needs to be invertible along the imaginary axis. By utilizing the decomposition into real and imaginary parts as in the previous section, one can verify that the imaginary and real parts of $\det B(s)$ are never simultaneously zero giving the desired result.

When considering the left-hand side of the equation and the associated difference operator D from (2.125) we make the substitution $y(\xi) = \phi'(\xi)$ to place the emphasis on the action of the difference operator

$$Dy_{\xi} = \sum_{j=1}^{N} B_j y_{\xi}(r_j).$$

In the context of the ansatz $y(\xi) = e^{\lambda\xi}$ we desire that the eigenvalues λ do not lie on the imaginary axis as in the above discussion. In the context of the ansatz $y(\xi) = \mu^{\xi}$, more commonly associated with difference operators, it is enough to check that the eigenvalues μ do not lie on the unit circle giving an exponential dichotomy. The eigenvalue correspondence among the differing ansatz is given by $\mu = e^{\lambda}$.

Additionally, instead of directly checking that the difference operator D exhibits exponential dichotomy, one can check the sufficient condition of diagonal dominance of the difference operator.

2.6.5 Fredholm Operator of Index 0

The previous sections have developed the step involved with satisfying the conditions of the Fredholm Alternative Theorem and yielding the fact that $\Lambda_{c_0,D,L}$ is a Fredholm operator. The Fredholm Index can now be computed, and it is computed via the Spectral Flow Property which tracks the net number of eigenvalues crossing the imaginary axis as the homotopy parameter ρ varies between 0 and 1. To verify that this index is zero, the homotopy Λ linking the characteristic matrices at $\pm\infty$ can be shown to have no eigenvalues lying on the imaginary axis for any $\rho \in [0,1]$, where

$$\Lambda(\rho) = (1-\rho)\Lambda_{c_0,D_-,L_-} + \rho\Lambda_{c_0,D_+,L_+}. \tag{2.130}$$

2.6.6 Isomorphism

This section contains parallels to section 3 of [26] and to [32]. Consider the nonlinear map $\mathscr{G} : W^{1,\infty} \times \mathbb{R} \times V \to L^\infty$ defined as

$$\mathscr{G}(\phi, c, \alpha) = -c \sum_{j=1}^{N} B_j(\alpha)\phi'(\xi + r_j) - \sum_{j=1}^{N} A_j(\alpha)\phi(\xi + r_j) + \sum_{j=1}^{N} C_j(\alpha) f(\phi(\xi + r_j)), \tag{2.131}$$

where V is an open subset of $\mathbb{R}$. Any root (ϕ_0, c_0, α_0) of the map $\mathscr{G}$ satisfying the boundary conditions (2.116) corresponds, for $\alpha = \alpha_0$, to the traveling wave solution pair (ϕ_0, c_0) of the nonlinear equation (2.114). If the profile ϕ_0 satisfies the equation, then any other translate satisfies the equation as well. Within the context of Newton's method, we may wish to find a particular translate. If ϕ_0 satisfies the phase condition $\phi_0(0) = a$, then the phase condition may be preserved if the profile updates ψ satisfy $\psi(0) = 0$. From this fact, we ask that $\psi \in W_0^{1,\infty}$. Let D_1 denote the Frechet derivative with respect to the first argument. Then, the Fréchet derivative of $\mathscr{G}$ with respect to the first and second arguments preserving phase is the linear operator $D_{1,2}\mathscr{G}(\phi_0, c_0, \alpha_0)$: $W_0^{1,\infty} \times \mathbb{R} \to L^\infty$ given by

$$D_{1,2}\mathscr{G}(\phi_0, c_0, \alpha_0)(\psi, b)(\xi) = -b \sum_{j=1}^{N} B_j(\alpha_0)\phi'(\xi + r_j) + (\Lambda_{c_0,D,L}\psi)(\xi), \tag{2.132}$$

where the linearization about ϕ_0 is $\Lambda_{c_0,D,L}$, and the first term is due to the derivative with respect to the wave speed c.

We consider linear operators $\Lambda_{c_0,D,L}$ with nontrivial kernels; specifically, we consider the cases when the kernel is of dimension one for scalar equations or of dimension d for systems of d equations. We note, in the scalar case, that ϕ_0' is always a solution to the homogeneous equation $\Lambda_{c_0,D,L}\psi = 0$ seen by direct differentiation of both sides of the nonlinear equation (2.114). With this setup, an application of the implicit function theorem may require an adjustment, particularly when considering a system. We will perform continuation starting with solutions that give rise to linear operators of the above type. Before we present an assumption we wish our solutions to satisfy, we make the following identifications in preparation for d-dimensional systems:

$$
\begin{aligned}
X &= W^{1,\infty} \oplus \cdots \oplus W^{1,\infty} \\
Y &= W^{1,2} \oplus \cdots \oplus W^{1,2} \\
Z &= L^2 \oplus \cdots \oplus L^2
\end{aligned}
\tag{2.133}
$$

and

$$
Y_0 = W_0^{1,2} \oplus W^{1,2} \oplus \cdots \oplus W^{1,2}. \tag{2.134}
$$

In the following, we are considering a system that decouples into d identical copies of a scalar equation:

Assumption B. *Assume that for the given system of equations (2.131) the system decouples into d identical copies of the scalar equation when* $\alpha = 0$*:*

$$
-c\phi'(\xi) - \sum_{j=1}^{N} A_j \phi(\xi + r_j) + f(\phi(\xi)) = 0 \tag{2.135}
$$

In particular, we are utilizing results from Mallet-Paret [32] about the nature of the linearized operator $\Lambda_{c_0,L}$ for each scalar equation, and in order to so, we will make the following assumption about each decoupled equation when $\alpha = 0$:

Assumption C. *For each decoupled equation of the form (2.135), with linearized operator of the form*

$$(\Lambda_{c_0,L}x)(\xi) = -cx'(\xi) - \sum_{j=1}^{N} A_j(\xi)x(\xi + r_j) \tag{2.136}$$

assume that the limiting operators $A_{j\pm} = \lim_{\xi\to\pm\infty} A_j(\xi)$ *satisfy the following condition*

$$A_{\Sigma_\pm} := \sum_{j=1}^{N} \tilde{A}_{j\pm} < 0 \tag{2.137}$$

with the additional restrictions that $0 < \tilde{A}_j(\xi)$ *for* $j > 1$.

We have collected some results for the scalar equation proved in [32], [43] in the following theorem:

Theorem 2.6.1. *For a scalar equation of the form (2.135) and linearized operator* $\Lambda_{c_0,L}$ *satisfying Assumption C, we have the following*

1. *A monotone increasing solution* $\phi \in L^\infty$ *exists to (2.135), with unique wave speed* c *and unique, up to translation, profile* ϕ.

2. *The dimension of the kernel of* $\Lambda_{c_0,L}$ *and consists of span*$\{\phi_0'\}$*:*

$$\dim \mathscr{K}(\Lambda_{c_0,L}) = 1, \quad \mathscr{K}(\Lambda_{c_0,L}) = span\{\phi_0'\}. \tag{2.138}$$

3. *The linear operator has Fredholm index 0*

$$\operatorname{ind}(\Lambda_{c_0,L}) = 0. \tag{2.139}$$

4. *The dimension of kernel of the adjoint is one and consists of the span of strictly positive element* $\phi^* \in L^\infty$

$$\dim \mathscr{K}(\Lambda^*_{c_0,L}) = 1. \tag{2.140}$$

We note, from the Fredholm Alternative, the kernel $\mathscr{K}(\Lambda_{c_0,L})$ and kernel of the adjoint $\mathscr{K}(\Lambda^*_{c_0,L})$ are independent of p, that is they are subsets of $W^{1,p}$ for all p. Hence, the Fredholm index is fixed for all p. Although the solution ϕ to (2.131) lives in $W^{1,\infty}$, we seek perturbations ψ to ϕ_0 that live $W^{1,2}$, thus giving a solution $\phi(\alpha) = \psi(\alpha) + \phi_0$. Since the Fredholm index and kernel of the linear operator are independent of p, we are free to apply the linear theory in $W^{1,2}$.

As an immediate consequence of Assumptions B and C and Theorem 2.6.1 we have

Proposition 2.6.2. *Assume that nonlinear equation (2.114) decouples as in Assumption B, and for a traveling wave solution* $(\phi_0, c_0) = (\phi_{1,0}, \ldots, \phi_{d,0}, c_0)$ *the linear system (2.124) satisfies Assumption C, then the following condition holds for* $\psi \in Y$ *and* $b \in \mathbb{R}$

$$D_{1,2}\mathscr{G}(\phi_0, c_0, 0)(\psi, b)(\xi) = 0 \quad \Leftrightarrow \quad b = 0 \text{ and } \psi(\xi) = K\phi_0'(\xi), \tag{2.141}$$

for some diagonal $K \in \mathbb{R}^{d\times d}$.

Proposition 2.6.3. *For* α_0*, suppose* $(\phi_0, c_0) \in X \times \mathbb{R}$ *is a traveling wave solution to (2.131) that satisfies Assumptions B and C with the associated linear operator* $\Lambda_{c_0,D,L}$*. Then,*

(i) In the scalar case, $D\mathscr{G}_{1,2}(\phi_0, c_0, 0)$ *is an isomorphism from* $W_0^{1,\infty} \times \mathbb{R}$ *onto* L^∞*.*

(ii) In the case of a system ($d > 1$)*, the dimension of the kernel*

$$\dim \mathscr{K}(D_{1,2}\mathscr{G}(\phi_0, c_0, 0)) = d - 1, \tag{2.142}$$

and the codimension of the range $\mathscr{R}(D\mathscr{G}_{1,2})$ is $d-1$ dimensional. Let $A := D\mathscr{G}_{1,2}(\phi_0, c_0, 0)$: $Y_0 \to Z$, then the following projections under the setup of the Lyapunov-Schmidt can be made

$$U : Y \to Y, \qquad Y_U = UY = \mathscr{N}(A), \tag{2.143}$$

$$E : Z \to Z, \qquad Z_E = EZ = \mathscr{R}(A). \tag{2.144}$$

In addition, the complementary spaces are given by $Y_{I-U} = (I-U)Y = \mathscr{N}(A)^{\perp}$ and $Z_{I-E} = (I-E)Z = \mathscr{R}(A)^{\perp}$. The linear operator A has a right inverse $K : Z_E \to Y_{I-U}$

$$AK = I, \qquad \text{on } Z_E$$

$$KA = I - U, \qquad \text{on } Y,$$

and A is an isomorphism from $Y_{I-U} \to Z_E$.

Proof. Scalar case (d=1): from Theorem 2.6.1, the dimension of the kernel $\dim(\mathscr{K}(\Lambda_{c_0,D,L}) = 1$ and $\text{ind}(\Lambda_{c_0,D,L}) = 0$. As a result, $\text{span}\{\phi_0'(\xi)\} = \mathscr{K}(\Lambda_{c_0,D,L}) \subset W^{1,\infty}$. Since ϕ is monotone increasing, $\phi_0'(0) > 0$ implying $\phi_0' \notin \mathscr{K}(\Lambda_{c_0,D,L})$ as a map from $W_0^{1,\infty} \times \mathbb{R} \to L^\infty$. Consequently, $\dim \mathscr{K}(D_{1,2}\mathscr{G}(\phi_0, c_0, \alpha_0)) = 0$ in the space $W_0^{1,\infty} \times \mathbb{R}$.

To conclude, we show that $D_{1,2}\mathscr{G}(\phi_0, c_0, 0))$ is onto. When $\alpha = \alpha_0$, the sum $\sum_{j=1}^{N} B_j(\alpha_0)\phi_0'(\xi + r_j)$ reduces to $\phi_0'(\xi)$, and it suffices to show $\phi_0'(\xi) \notin \mathscr{R}(\Lambda_{c_0,D,L})$. But, as in [32] or [26], it follows that the element $p^* \in \mathscr{K}(\Lambda^*_{c_0,D,L})$ is strictly positive. But $\phi_0'(\xi) > 0$ and the Fredholm Alternative gives $\int \phi_0' \, p^* d\xi \neq 0$ implying that $\phi_0' \notin \mathscr{R}(\Lambda_{c_0,D,L})$. It now follows that $D_{1,2}\mathscr{G}(\phi_0, c_0, 0))$ is onto giving an isomorphism as a map $W_0^{1,\infty} \times \mathbb{R} \to L^\infty$.

System case ($d > 1$): we refer to Section 3.1 were the case $d = 2$ has been examined in detail. We note that it follows from Proposition 2.6.2 and the phase restriction present in Y_0 that $\dim \mathscr{K}(D_{1,2}\mathscr{G}(\phi_0, c_0, 0)) = d - 1$. □

Remark 2.6.4. *Although the system decouples into d identical copies of the scalar equation, each component profile ϕ_j may potentially have a different translate β_j, i.e. $\phi(\xi + \beta_j) = a$. We require that the translate of ϕ_1 stay fixed with $\psi_1 \in W_0^{1,2}$, but the other profiles are free to translate.*

2.6.7 Local Continuation in Parameter α

Local continuation in α relies upon the implicit function theorem, and, in particular, the fact that $D_{1,2}\mathscr{G}(\phi_0, c_0, \alpha_0)$ is an isomorphism. The implicit function theorem as stated in Ambrosetti and Prodi [2] is presented below:

Theorem 2.6.5. ***(Implicit Function Theorem)*** *Let $F \in C^k(\Lambda \times U, Y)$, $k \geq 1$, where Y is a Banach space and Λ (resp. U) is an open subset of Banach Space T (resp. X). Suppose that $F(\lambda^*, u^*) = 0$ and that $F_u(\lambda^*, u^*)$ is an invertible linear map from X onto Y.*

Then there exist neighborhoods Θ of λ^* in T and U^* of u^* in X and a map $g \in C^k(\Theta, X)$ such that

1. *$F(\lambda, g(\lambda)) = 0$ for all $\lambda \in \Theta$,*

2. *$F(\lambda, u) = 0$, $(\lambda, u) \in \Theta \times U^*$, implies $u = g(\lambda)$,*

3. *$g'(\lambda) = -[F_u(p)]^{-1} \circ F_\lambda(p)$, where $p = (\lambda, g(\lambda))$ and $\lambda \in \Theta$.*

By translating the known solution (ϕ_0, c_0) for α_0 to the origin by the map $G : W_0^{1,\infty} \times \mathbb{R} \to L^\infty$

$$G(\phi, c, \alpha) = \mathscr{G}(\phi + \phi_0, c, \alpha), \tag{2.145}$$

the implicit function theorem can be applied wherein $\phi \in W_0^{1,\infty}$. We later use this translation of the origin when applying Lyapunov-Schmidt in Section 3.1.

2.7 Example 1: Model Problems

2.7.1 Models Considered

In this section we consider particular cases of (2.113). After linearizing about a known solution ϕ_0 we obtain the linearized equations as in (2.124)

$$-c_0 \sum_{j=1}^{N} B_j(\alpha_0)\phi'(\xi + r_j) = \sum_{j=1}^{N} \tilde{A}_j(\xi;\alpha_0)\phi(\xi + r_j)$$

with the associated linear operator $\Lambda_{c_0,D,L}$ as in (2.126). We consider the following cases, where the nonlinearity f refers to the cubic (2.117):

Model 1. $C_j(\alpha) = B_j(\alpha)$ as in (2.114) with the coefficients

j	1	2	3
r_j	0	1	-1
$B_j(\alpha_0)$	$1-2\alpha_0$	α_0	α_0
$\tilde{A}_j(\xi;\alpha_0)$	$-2 - B_1(\alpha_0)\beta(\xi)$	$1 - B_2(\alpha_0)\beta(\xi+1)$	$1 - B_3(\alpha_0)\beta(\xi-1)$

Model 1 is based on considering the implicitly defined problem posed on a lattice

$$V_n + \alpha(V_{n+1} - 2V_n + V_{n-1}) = (U_{n+1} - 2U_n + U_{n-1}), \text{ for } n \in \mathbb{R}, \tag{2.146}$$

where $V_n = \dot{U}_n + f(U_n)$. By subsequently imposing a traveling wave ansatz Model 1 results.

Model 2. $C_1(\alpha) = 1, C_j(\alpha) = 0$ for $j \neq 1$ as in (2.115) with $\tau_2 > \tau_1 > 0$ and

j	1	2	3	4	5
r_j	0	τ_1	$-\tau_1$	τ_2	$-\tau_2$
$B_j(\alpha_0)$	$1-2\alpha_0$	α_0	α_0	0	0
$\tilde{A}_j(\xi;\alpha_0)$	$-2-\beta(\xi)$	0	0	1	1

Model 2a. This subcase will consider the values $\tau_1 = 1$ and $\tau_2 = 2$. This subcase is a result of considering the following spatially discrete equation

$$\dot{U}_n + \alpha(\dot{U}_{n-1} - 2\dot{U}_n + \dot{U}_{n+1}) = (U_{n-2} - 2U_n + U_{n+2}) - f(U_n). \tag{2.147}$$

Recall, in the cases considered above, the limiting equations at $\pm\infty$ are given by considering the limiting values of $\beta(\xi)$ as in (2.120).

2.7.2 Characteristic Matrix

The characteristic matrices for the limiting equations given in (2.128) are calculated below the two models. We utilize the identity $2(\cosh(\lambda)-1) = e^{\lambda} - 2 + e^{-\lambda}$ to simplify the characteristic matrices. In particular, $\Delta_1 e^{\lambda\xi} = e^{\lambda\xi}(e^{\lambda} - 2 + e^{-\lambda})$, where Δ_1 denotes the discrete Laplacian (3.62).

Model 1.

$$\begin{aligned}&\Delta_{c_0,D_\pm,L_\pm}(\lambda)\\ &= (-c_0\lambda)[1 + 2\alpha_0(\cosh(\lambda)-1)] - 2(1-\alpha_0\beta_\pm)(\cosh(\lambda)-1) + \beta_\pm.\end{aligned} \tag{2.148}$$

Model 2.

$$\Delta_{c_0,D_\pm,L_\pm}(\lambda)$$
$$= (-c_0\lambda)[1+2\alpha_0(\cosh(\tau_1\lambda)-1)] - 2(\cosh(\tau_2\lambda)-1)+\beta_\pm. \quad (2.149)$$

2.7.3 Asymptotically Hyperbolic

By way of contradiction, suppose $\Delta_{c_0,D_\pm,L_\pm}(\lambda) = 0$ for some $\lambda = i\eta$ with $\eta \in \mathbb{R}$. It will be shown that the real and imaginary parts of the characteristic equation can never be simultaneously zero. Recall, the for the cubic $\beta(\xi)$ takes the limiting values of $\beta_+ = 1-a$ and $\beta_- = a$.

Model 1. Suppose $\Delta(i\eta) = 0$

$$0 = (-ic_0\eta)[1+2\alpha_0(\cos(\eta)-1)] - 2(1-\alpha_0\beta_\pm)(\cos(\eta)-1)+\beta_\pm.$$

Isolating $\cos(\eta)$ gives

$$\begin{aligned}\cos(\eta) &= \frac{[2(1-\alpha_0\beta_\pm)+\beta_\pm] + i(2\alpha_0-1)c_0\eta}{2(1-\alpha_0\beta_\pm)+i2\alpha_0c_0\eta} \\ &:= \frac{a+bi}{d+ei},\end{aligned}$$

where $a = 2(1-\alpha_0\beta_\pm)+\beta_\pm$, $b = (2\alpha_0-1)c_0\eta$, $d = 2(1-\alpha_0\beta_\pm)$, and $e = 2\alpha_0c_0\eta$. Since $\cos\eta$ is a real-valued solution exists if the fraction $\frac{a+bi}{d+ei}$ is real-valued . By multiplying the numerator and denominator by the conjugate $d-ei$ it can be seen that the right-hand side is real-valued only if $db-ae = 0$. Further, suppose $db = ae$, and we find

$$db = ae \quad \Leftrightarrow \quad 2c_0\eta = 0.$$

By assumption we have $c_0 \neq 0$. If $\eta = 0$, then it is necessary for $1 = \cos(0) = \frac{a}{d}$. But, $a = d + \beta_{\pm}$, and since both $\beta_{\pm} > 0$ the equation is asymptotically hyperbolic regardless of α_0.

Model 2. If $\Delta(i\eta) = 0$, then the system of equations given by the real and imaginary parts that is satisfied

$$2(\cos(\tau_2\eta) - 1) = \beta_{\pm}, \tag{2.150}$$

$$c_0\eta(1 + 2\alpha_0(\cos(\tau_1\eta) - 1)) = 0. \tag{2.151}$$

Since $\beta_{\pm} > 0$, (2.150) cannot be satisfied. As a result, Model 2 is asymptotically hyperbolic regardless of α_0.

2.7.4 Diagonal Dominance of B(s)

Recall, the left-hand side difference operator D gives rise to the matrix-valued function $B(s) = \sum_{j=1}^{3} B_j(\alpha_0)e^{sr_j}$ as in (2.24). We demonstrate diagonal dominance along the imaginary axis.

Model 1. Using the coefficient values of B_j's as defined in section 2.7.1 we obtain the following norm calculations

$$||B_1^{-1}(\alpha_0)||^{-1} = |1 - 2\alpha_0|,$$

$$\sum_{j=2}^{3} ||B_j(\alpha_0)|| = 2|\alpha_0|.$$

The inequality $|1-2\alpha_0| > 2|\alpha_0|$ is satisfied if (assuming $\alpha_0 > 0$)

$$\alpha_0 < \frac{1}{4}. \tag{2.152}$$

Model 2. The coefficients of the left-hand side are identical to the coefficients of Model 1, and although the shifts differ, the computation along the imaginary axis is identical to above. The condition $\alpha_0 < 1/4$ gives diagonal dominance.

2.7.5 Exponential Dichotomy

Since the corresponding shifts of the nonzero coefficients of D are rationally related it is possible to rewrite the difference operator D as a first order system of equations.

Model 1. Assuming $\alpha > 0$, we take the difference operator D

$$(Dy)(t) := (1-2\alpha)y(t) + \alpha y(t+1) + \alpha y(t-1),$$

and rewrite it as a first order system

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2-\frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{2.153}$$

where $x(t) = y(t-1)$. The resulting characteristic equation is

$$\left(\lambda^2 - \left(2 - \frac{1}{\alpha}\right)\lambda + 1\right) = 0. \tag{2.154}$$

The associated eigenvalues are

$$\lambda_{\pm} = -(\frac{1}{2\alpha} - 1) \pm \frac{1}{2\alpha}\sqrt{1-4\alpha}, \text{ where } \lambda_+ = \frac{1}{\lambda_-}. \tag{2.155}$$

For small positive α the eigenvalues $\lambda_{\pm}$ do not lie on the unit circle, and in particular

$$\lambda_{-} = \frac{-1}{\alpha} + O(1), \quad \lambda_{+} = -\alpha + O(\alpha^2). \tag{2.156}$$

More specifically, one can find that

$$|\lambda_{\pm}| \neq 1, \quad \text{if } 0 \leq \alpha_0 < 1/4, \tag{2.157}$$

and also $|\lambda_{\pm}| = 1$ if $\alpha_0 \geq 1/4$. The eigenvalue, eigenvector pairs are

$$\left(\lambda_{-}, \begin{bmatrix} \lambda_{+} \\ 1 \end{bmatrix}\right), \quad \left(\lambda_{+}, \begin{bmatrix} \lambda_{-} \\ 1 \end{bmatrix}\right). \tag{2.158}$$

The same result and multiplicity is also obtained if the eigenvalues are computed by substituting the difference equation ansatz $y(t) = \lambda^t$

$$\alpha\lambda^{-1} + (1-2\alpha) + \alpha\lambda = 0. \tag{2.159}$$

Assuming that $\lambda \neq 0$, we obtain the modified characteristic equation

$$\frac{1}{2\alpha\lambda}\left(\lambda^2 + \left(\frac{1}{\alpha} - 2\right)\lambda + 1\right) = 0 \tag{2.160}$$

in which the expression within the outer parentheses is the same expression obtained in (2.154).

Model 2. The difference operator D under consideration is

$$(Dy)(t) = (1-2\alpha)y(t) + \alpha y(t+\tau_1) + \alpha y(t-\tau_1).$$

and rewriting it as a first order system

$$\begin{pmatrix} x(t+\tau_1) \\ y(t+\tau_1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2-\frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{2.161}$$

where $x(t) = y(t-\tau_1)$. The coefficient matrix in (2.161) is the same that was obtained in (2.153), thereby giving the same characteristic equation and the same eigenvalues.

By direct substitution of the difference equation ansatz $y(t) = \lambda^t$ one obtains the equation

$$\lambda^t(\alpha\lambda^{-\tau_1} + (1-2\alpha) + \alpha\lambda^{\tau_1}) = 0. \tag{2.162}$$

Making the identification $\mu = \lambda^{\tau_1}$ one can rewrite the characteristic equation to obtain the same characteristic equation as (2.160), but in μ. One can find that the eigenvalues $|\lambda_\pm| \neq 1$ under the same restriction as in (2.157). While the magnitude of the eigenvalues may differ from (2.161) to (2.162), this discrepancy can be resolved by rescaling time. For these scalar equations the restrictions placed on α giving rise to an exponential dichotomy are equivalent to the restrictions on α to enforce diagonal dominance.

2.7.6 Computation of Fredholm Index

Model 1. Let $\Lambda(\rho)$ be the linear homotopy between the limiting operators at $\mp\infty$:

$$\Lambda(\rho) = (1-\rho)\Lambda_{c_0,D_-,L_-} + \rho\Lambda_{c_0,D_+,L_+}. \tag{2.163}$$

We show that no eigenvalues cross the imaginary axis as the homotopy parameter varies between 0 and 1; this verifies that the index of the Fredholm operator is 0. In a similar manner to section (2.7.3) suppose $\Delta_\rho(i\eta) = 0$, where $\Delta_\rho(\lambda)$ is the associated characteristic matrix to $\Lambda(\rho)$. Additionally, for the following calculation let $K(\lambda) = 2(\cosh(\lambda) - 1)$, and recall $\beta_- = a$, $\beta_+ = 1 - a$ from (2.120):

$$
\begin{aligned}
0 &= (1-\rho)\Delta_{c_0,D_-,L_-}(i\eta) + \rho\Delta_{c_0,D_+,L_+}(i\eta) \\
&= -c_0 i\eta[1 + \alpha_0 K(i\eta)] + (\alpha_0[(1-\rho)\beta_- + \rho\beta_+] - 1)K(i\eta) + ((1-\rho)\beta_- + \rho\beta_+).
\end{aligned}
$$

Isolating $K(i\eta)$ we find

$$
\begin{aligned}
K(i\eta) &= \frac{-[(1-\rho)\beta_- + \rho\beta_+] + ic_0\eta}{\alpha_0[(1-\rho)\beta_- + \rho\beta_+] - 1 - i\alpha_0 c_0\eta} \\
&:= \frac{b + id}{e + ig},
\end{aligned}
$$

where $b = -[(1-\rho)\beta_- + \rho\beta_+]$, $d = c_0\eta$, $e = \alpha_0[(1-\rho)\beta_- + \rho\beta_+] - 1$, and $g = -\alpha_0 c_0\eta$. Following the same procedure as in the section (2.7.3), we multiply the the numerator and denominator of the right-hand side by the conjugate $e - ig$. If the right-hand side is real-valued the imaginary part must be zero, i.e. $de - bg = 0$. Under the restriction that $\alpha_0 < 1/4$ and $a, \rho \in [0,1]$, $de - bg = 0$ if and only if $c_0\eta = 0$. We assume that we are linearizing about a traveling wave, therefore $c_0 \neq 0$ by assumption. If $\eta = 0$, then it must true that

$$
0 = \frac{-[(1-\rho)a + \rho(1-a)]}{\alpha_0[(1-\rho)a + \rho(1-a)] - 1},
$$

in which case the relation is satisfied if and only if $\rho = \frac{-a}{1-2a}$; however, for $a \in (0,1)$ it follows that $\rho \notin [0,1]$.

Model 2. In a similar manner to Model 1, one can obtain a similar result for the homotopy connecting the limiting operators for Model 2 containing the presence of the terms $K(i\tau_1\eta)$ and $K(i\tau_2\eta)$

$$\begin{aligned}0 &= (1-\rho)\Delta_{c_0,D_-,L_-}(i\eta)+\rho\Delta_{c_0,D_+,L_+}(i\eta)\\ &= -c_0 i\eta[1+\alpha_0 K(i\tau_1\eta)]-K(i\tau_2\eta)+[(1-\rho)\beta_- +\rho\beta_+].\end{aligned}$$

Bringing the real terms to the left-hand side and the imaginary terms on the right-hand side we obtain

$$[(1-\rho)a+\rho(1-a)]-K(i\tau_2\eta)=ic_0\eta[1+\alpha_0 K(i\tau_1\eta)].$$

The right-hand side has an imaginary part if $\eta\neq 0$ since $K(i\cdot)$ is real-valued, and the fact that $\alpha_0<1/4$ with $\max_\eta|K(i\tau_1\eta)|=4$ bounds the right-hand side away from zero. If $\eta=0$, then a similar argument to Model 1 applies to show that the equation is satisfied when $\rho=\frac{1-a}{1-2a}\notin[0,1]$.

2.7.7 Isomorphism and Local Continuation

In Model 1 or Model 2 the left-hand side coefficients $B(\alpha)$ reduce to the identity when $\alpha=0$. In addition, the sum of the A coefficients is $A_{\Sigma_\pm}=-\beta_\pm<0$. It follows from Proposition (2.6.3) that $D\mathscr{G}(\phi_0,c_0,\alpha_0)$ is isomorphism.

Theorem 2.7.1. *For Model 1 or Model 2 the associated linearized operator (2.132) is Fredholm of Index 0 with a trivial kernel, via the restriction on the domain i.e. $\phi\in W_0^{1,\infty}$. Hence, given an initial solution (ϕ_0,c_0), the solution (ϕ_α,c_α) is given by the implicit function theorem for small α.*

Chapter 3

Ephaptic Coupling

The model considered by Binczak, Eilbeck, and Scott [11] and later by Bateman and Van Vleck [5] is an example of neutral type/implicitly defined. This model describes the electrical interaction between two parallel nerve fibers. In [5] the nonlinearity considered is the piecewise linear cubic which permits the direct application of Fourier transforms. We consider the system with the cubic-like nonlinearity (2.118), and note that similar calculations full cubic as the nonlinearity.

The parallel nerve fiber model is illustrative as to how implicitly defined equations arise. The resulting circuit model from [11] is reproduced in Figure 3.1. The cells from the two strands are staggered by a distance A and linked through the resistor R_0. The resulting equations after applying a mesh analysis and some simplification are

$$
\begin{aligned}
d(V_n^{(1)} - V_{n+1}^{(1)}) &= i_n^{(1)} + \alpha(A i_{n-1}^{(2)} + (1-A) i_n^{(2)}) && (3.1)\\
d(V_n^{(2)} - V_{n+1}^{(2)}) &= i_n^{(2)} + \alpha(A i_{n+1}^{(1)} + (1-A) i_n^{(1)}) &&\\
i_{n-1}^{(1)} - i_n^{(1)} &= RC\dot{V}_n^{(1)} + f(V_n^{(1)}) && (3.2)\\
i_{n-1}^{(2)} - i_n^{(2)} &= RC\dot{V}_n^{(2)} + f(V_n^{(2)}). &&
\end{aligned}
$$

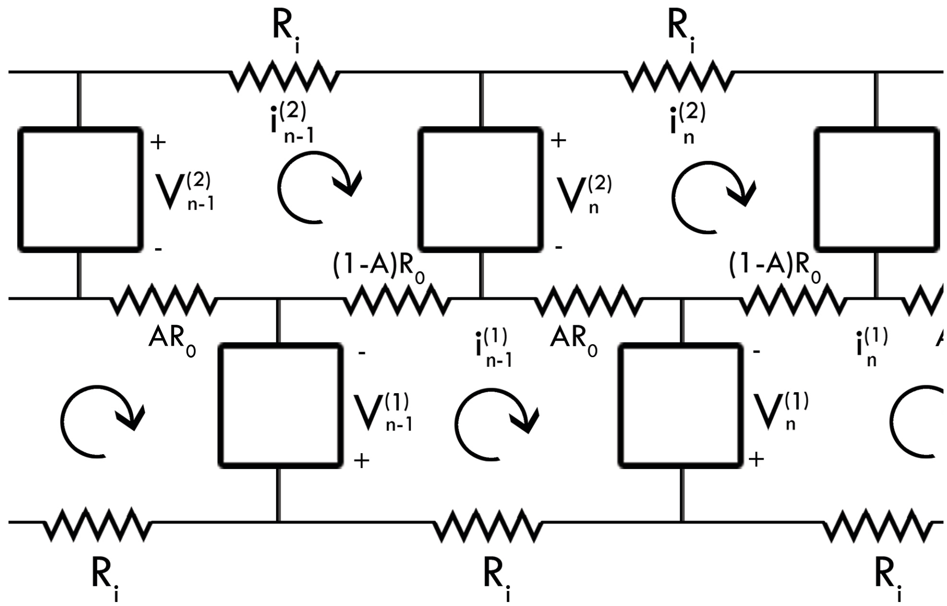


Figure 3.1: This is an example of a portion of a circuit that is an infinitely long chain with staggered top and bottom layers.

To obtain closed form differential equations we solve for $i_n^{(1)}$ and $i_n^{(2)}$ in (3.1) and then substitute, using translates as needed, into (3.2). The resulting equations become implicitly defined due to the presence of the translates $i_{n-1}^{(2)}$ and $i_{n+1}^{(1)}$ in (3.1). This can be seen by rewriting (3.1) as a system of equations for $i_n^{(1)}$ and $i_n^{(2)}$ in the form

$$\begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} i_n^{(1)} \\ i_n^{(2)} \end{bmatrix} = \begin{bmatrix} X_n^{(1)} \\ X_n^{(2)} \end{bmatrix}, \tag{3.3}$$

where Kirchoff's current law equations (3.2) are used to eliminate the translates $i_{n-1}^{(2)}$ and $i_{n+1}^{(1)}$

$$\begin{aligned} i_{n+1}^{(1)} &= i_n^{(1)} - RC\dot{V}_{n+1}^{(1)} - f(V_{n+1}^{(1)}) \\ i_{n-1}^{(2)} &= i_n^{(2)} + RC\dot{V}_n^{(2)} + f(V_n^{(2)}). \end{aligned} \tag{3.4}$$

As a result the translated derivative term $\dot{V}_{n+1}^{(1)}$ and $\dot{V}_{n}^{(2)}$ are concurrently introduced into the right-hand side $\begin{bmatrix} X_n^{(1)} \\ X_n^{(2)} \end{bmatrix}$, and the currents take the form

$$
\begin{aligned}
i_n^{(1)} &= g_1(\dot{V}_{n+1}^{(1)}, \dot{V}_n^{(2)}, V_n^{(1)}, V_{n+1}^{(1)}, V_n^{(2)}, V_{n+1}^{(2)}) \\
i_n^{(2)} &= g_2(\dot{V}_{n+1}^{(1)}, \dot{V}_n^{(2)}, V_n^{(1)}, V_{n+1}^{(1)}, V_n^{(2)}, V_{n+1}^{(2)}).
\end{aligned} \tag{3.5}
$$

Once the terms $\dot{V}_{n+1}^{(1)}$, $\dot{V}_n^{(2)}$ are introduced through manipulation the resulting equations are necessarily implicitly defined

$$
B_2 W_{n-1} + B_1 W_n + B_3 W_{n+1} = A_2 V_{n-1} + A_1 V_n + A_3 V_{n+1}, \tag{3.6}
$$

where $W_n = \dot{V}_n + f(V_n)$ and $V_n = [V_n^{(1)} \;\; V_n^{(2)}]^T$. The B_j and A_j coefficients depend on the parameters α and d and by setting $RC = 1$ the coefficients take the following form using $\kappa = \frac{1}{1-\alpha^2}$

$$
B_1 = I + \kappa \begin{pmatrix} \alpha^2 A & -\alpha A \\ -\alpha A & \alpha^2 A \end{pmatrix}, \;\; B_2 = \kappa \begin{pmatrix} 0 & \alpha A \\ 0 & -\alpha^2 A \end{pmatrix}, \;\; B_3 = \kappa \begin{pmatrix} -\alpha^2 A & 0 \\ \alpha A & 0 \end{pmatrix}, \tag{3.7}
$$

and the A_j coefficients are

$$
A_1 = \kappa d \begin{pmatrix} -2 & 2\alpha \\ 2\alpha & -2 \end{pmatrix}, \quad A_2 = A_3 = \kappa d \begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 \end{pmatrix}. \tag{3.8}
$$

By imposing the traveling wave ansatz $v_n = \phi(n-ct)$ with $\xi = n-ct$ the traveling wave equations reveal the implicitly defined mixed type model

$$\sum_{j=1}^{3} B_j \phi'(\xi + r_j) = \sum_{j=1}^{3} A_j \phi(\xi + r_j) - \sum_{j=1}^{3} B_j f(\phi(\xi + r_j)), \tag{3.9}$$

where $r_1 = 0, r_2 = -1$, and $r_3 = +1$.

Notice that if the strands are staggered by an amount $A = 0$ or the parameter $\alpha = 0$ then the translates $i_{n-1}^{(2)}$ and $i_{n+1}^{(1)}$ do not appear in (3.1) and the resulting equations are explicit.

Our approach to finding a solution to the 2-dimensional system is to perturb away from $\alpha_0 = 0$. When $\alpha_0 = 0$ the 2-dimensional systems decouples, and we can use results from the scalar case. When $\alpha = \alpha_0$, the matrix coefficients simplify to

$$B_1 = I, \quad B_2 = B_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.10}$$

$$A_0 = d \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_2 = A_1 = d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The parameter value $\alpha_0 = 0$ not only decouples the problem, but it eliminates the implicit nature of the problem by eliminating the coefficients B_2, B_3. After the equations decouple, what remains are two copies of the discrete Nagumo equation

$$\dot{u}_j = d(u_{j-1} - 2u_j + u_{j+1}) - f(u_j). \tag{3.11}$$

Imposing the traveling wave ansatz $u_n = \phi(n-ct)$ with $\xi = n-ct$ the traveling wave equation has the form

$$-c\phi'(\xi) = d(\phi(\xi-1) - 2\phi(\xi) + \phi(\xi+1)) - f(\phi(\xi)). \tag{3.12}$$

Zinner [43] and, more generally, Mallet-Paret [32] proved the existence of a traveling wave solution $\phi_0(\xi)$ with $c > 0$ for (3.12) with the cubic nonlinearity and the class of bistable nonlinearities, respectively, given a sufficiently large d.

Remark: our calculations for (3.12) are performed for the cubic-like nonlinearity (2.118), and we will utilize the fact that with the rational cubic-like nonlinearity (3.12) yields a closed form solution. We would also like to emphasize that the calculations of Chapter 3 are minimally changed if the cubic (2.117) is used.

The linearization about $\phi_0(\xi)$ with wave speed c_0 is

$$-c_0\phi' = d(\phi(\xi-1) - 2\phi(\xi) + \phi(\xi+1)) - \beta(\xi)\phi(\xi),$$

where $\beta(\xi) = f'(\phi_0(\xi))$. Let L be the linear functional $C([-1,1],\mathbb{R}) \to \mathbb{R}$

$$L(\xi;d)\psi = d(\psi_\xi(1) - 2\psi_\xi(0) + \psi_\xi(-1)) - \beta(\xi)\psi_\xi(0). \tag{3.13}$$

Finally, define the linear operator $\Lambda_{c_0,L}\colon W^{1,p} \to L^p$

$$\begin{aligned}(\Lambda_{c_0,L}\phi)(\xi) &= -c\phi'(\xi) - L(\xi;d)\phi_\xi \\ &= -c_0\phi'(\xi) - d(\phi(\xi-1) - 2\phi(\xi) + \phi(\xi+1)) + \beta(\xi)\phi(\xi). \end{aligned} \tag{3.14}$$

3.0.8 Characteristic Matrix

In the case that L_0 is the constant coefficient operator

$$L_0(\xi;d)\psi = d(\psi_\xi(1) - 2\psi_\xi(0) + \psi_\xi(-1)) - \beta_0\psi_\xi(0), \tag{3.15}$$

the associated characteristic matrix for Λ_{c_0,L_d^0} can be written as

$$\Delta_{c_0,L_0}(\lambda) = -c_0\lambda - 2d(\cosh\lambda - 1) + \beta_0. \tag{3.16}$$

3.0.9 Asymptotically Hyperbolic

Recall, to show the system is asymptotically hyperbolic, we need to show the characteristic equation for the limiting operators satisfy $\det\Delta_{c_0,L_\pm}(i\eta) \neq 0$, $\forall\eta\in\mathbb{R}$. Substituting $\lambda = i\eta$ into (3.16) and equating real and imaginary parts gives:

$$\begin{aligned} 0 &= ic_0\eta, \\ 0 &= 2d(\cos\eta - 1) - \beta_\pm. \end{aligned} \tag{3.17}$$

Since $c_0 > 0$ the term $ic_0\eta$ is nonzero when $\eta \neq 0$. When $\eta = 0$, the bottom equation cannot be satisfied for $0 \neq -\beta_\pm$, where $\beta_\pm > 0$ from (2.123). Hence, this system is asymptotically hyperbolic for the detuning parameter $a \in (0,1)$.

3.0.10 Diagonal Dominance of $B(s)$

The quantity $\|B_1^{-1}\|^{-1}$ can be computed by finding the smallest eigenvalue of

$$B_1(\alpha) = \begin{pmatrix} 1+\alpha^2\kappa A & -\alpha\kappa A \\ -\alpha\kappa A & 1+\alpha^2\kappa A \end{pmatrix}.$$

The two eigenvalues of B_1 are $\lambda_1 = 1+\alpha\kappa A(\alpha+1)$ and $\lambda_2 = 1+\alpha\kappa A(\alpha-1)$. Assuming $\alpha > 0$ is small and $A > 0$, then $\lambda_1 > \lambda_2$, and

$$||B_2(\alpha)|| = ||B_3(\alpha)|| = \sqrt{(\alpha\kappa A)^2 + (\alpha^2\kappa A)^2} = \alpha\kappa A\sqrt{1+\alpha^2}. \tag{3.18}$$

Recall, $\kappa = 1/(1-\alpha^2)$ and diagonal dominance is achieved if

$$\lambda_2 > 2\alpha\kappa A\sqrt{1+\alpha^2} \Leftrightarrow 1 > \alpha\kappa A\left((1-\alpha)+2\sqrt{1+\alpha^2}\right) =: h(\alpha, A). \tag{3.19}$$

The parameter regime giving diagonal dominance is depicted in (3.2).

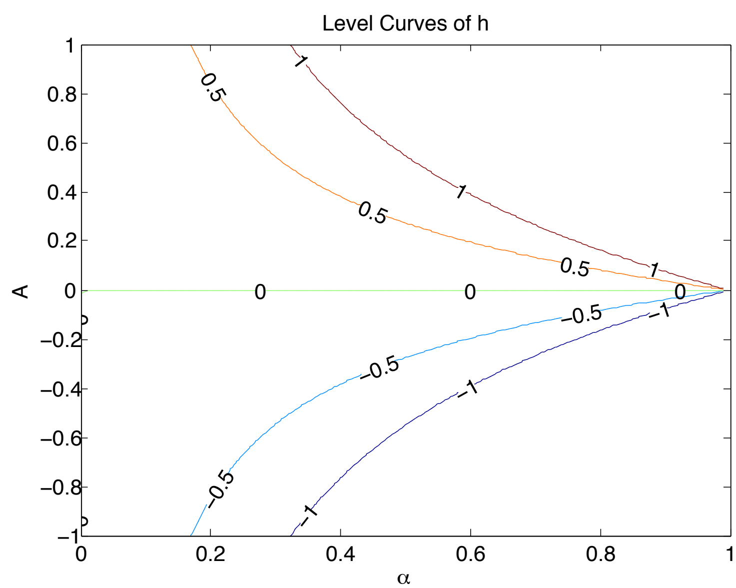


Figure 3.2: Diagonal dominance is achieved under the condition that that $h < 1$.

3.0.11 Exponential Dichotomy

Using the coefficients (3.7) and examining the inhomogeneous difference equation for bounded f_n

$$B_2 W_{n-1} + B_1 W_n + B_3 W_{n+1} = f_n \tag{3.20}$$

with the identification $W_n = (X_n, Y_n)^T$ and $\gamma = \alpha\kappa A$, the equations become

$$\begin{aligned} \gamma Y_{n-1} + (1+\alpha\gamma)X_n - \gamma Y_n - \alpha\gamma X_{n+1} &= f_{1,n} \\ -\alpha\gamma Y_{n-1} + (1+\alpha\gamma)Y_n - \gamma X_n + \gamma X_{n+1} &= f_{2,n}. \end{aligned} \tag{3.21}$$

In the form of a first order map the equations become

$$\begin{pmatrix} -\alpha\gamma & -\gamma \\ \gamma & 1+\alpha\gamma \end{pmatrix} \begin{pmatrix} X_{n+1} \\ Y_n \end{pmatrix} = - \begin{pmatrix} 1+\alpha\gamma & \gamma \\ -\gamma & -\alpha\gamma \end{pmatrix} \begin{pmatrix} X_n \\ Y_{n-1} \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_n . \tag{3.22}$$

Define M_1, M_2 to be the coefficient matrices

$$M_1(\alpha) = \begin{pmatrix} -\alpha\gamma & -\gamma \\ \gamma & 1+\alpha\gamma \end{pmatrix}, \quad M_2(\alpha) = \begin{pmatrix} 1+\alpha\gamma & \gamma \\ -\gamma & -\alpha\gamma \end{pmatrix}, \tag{3.23}$$

and we note that $M_1^{-1}(\alpha) = (1/\det M_2(\alpha))M_2(\alpha)$ as long as $\alpha, A > 0$. If α or A is zero, then the difference equation reduces to the explicit case and gives $(X_n, Y_n)^T = (f_1, f_2)_n^T$. Assuming that $\alpha, A \neq 0$ we have

$$\begin{pmatrix} X_{n+1} \\ Y_n \end{pmatrix} = \frac{-1}{\det M_2} M_2^2 \begin{pmatrix} X_n \\ Y_{n-1} \end{pmatrix} + \frac{1}{\det M_2} M_2 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_n , \tag{3.24}$$

with

$$\det(M_2) = -\gamma(\alpha - \gamma + \alpha^2\gamma) =: -b. \tag{3.25}$$

Letting $M = M_2^2$, the entries of the matrix M are

$$\begin{aligned} M_{11} &= 1 + \alpha\gamma + b, \quad M_{12} = \gamma, \\ M_{21} &= -\gamma, \qquad\qquad M_{22} = -\gamma^2(1-\alpha^2), \end{aligned} \tag{3.26}$$

and the eigenvalues of $(1/b)M$ are

$$\begin{aligned} \lambda_{\pm} \quad &= \tfrac{Tr(M) \pm \sqrt{(Tr(M)^2 - 4\det(M)}}{2b}, \text{ with } \det(M) = b^2, \\ Tr(M) = 1 + 2b, &\quad Tr(M)^2 - 4\det(M) = 1 + 4b. \end{aligned} \tag{3.27}$$

Noting that $\det((1/b)M) = 1$, simplification gives

$$\lambda_{\pm} = 1 + \frac{1}{2b}(1 \pm \sqrt{1+4b}), \quad \text{with } \lambda_{+} = \frac{1}{\lambda_{-}}. \tag{3.28}$$

Taking limits as α or A go to zero, implies $\gamma \to 0$ and $b \to 0$ with

$$\lim_{b\to 0} \lambda_{+} = +\infty, \quad \lim_{b\to 0} \lambda_{-} = 0. \tag{3.29}$$

3.0.12 Λ is a Fredholm Operator of Index 0

The previous calculations and the Fredholm Alternative Theorem demonstrate that Λ_{c_0,L_d} is a Fredholm operator. The linear homotopy between the limiting operators at $\pm\infty$ is given below

$$\Lambda(\rho) = (1-\rho)\Lambda_{c_0,L_-} + \rho\Lambda_{c_0,L_+}. \tag{3.30}$$

In particular, to show that the system is Fredholm Index 0 we will show that $\Lambda(\rho)$ has zero eigenvalues crossing the imaginary axis. Let $\Delta_\rho(\lambda)$ be the associated characteristic matrix to Λ_ρ. Set $\lambda = i\eta$, and suppose $\Delta_\rho(i\eta) = 0$:

$$0 = [-c_0 i\eta - 2d(\cos\eta - 1)] + [(1-\rho)\beta_- + \rho\beta_+]. \tag{3.31}$$

By splitting into real and imaginary parts, we have the equations

$$\begin{aligned} 0 &= -ic_0\eta, \\ 0 &= -2d(\cos\eta - 1) + [(1-\rho)\beta_- + \rho\beta_+]. \end{aligned} \tag{3.32}$$

The first equation can only be satisfied for $\eta = 0$. For $\eta = 0$ the second equation is satisfied when $\rho = \frac{-\beta_-}{\beta_+ - \beta_-}$. From (2.123) we know that $\beta_- = d\gamma + cb$ and $\beta_- = d\gamma - cb$. Further, we know that $d\gamma > cb$ for $a \in (0,1)$ which gives

$$\rho = \frac{-\beta_-}{\beta_+ - \beta_-} = \frac{-(d\gamma + cb)}{-2cb} = \frac{1}{2}(1 + \frac{d\gamma}{cb}) > 1,$$

which gives the fact that no eigenvalues cross the imaginary axis for $\rho \in [0,1]$ and $\mathrm{ind}(\Lambda_{c_0,L_d}) = 0$.

3.0.13 Isomorphism and Local Continuation

The existence and stability of solutions for this model will be discussed in the next sections.

3.1 Lyapunov-Schmidt

Suppose (φ_0, c_0) is a solution to the scalar problem (3.12) for $\alpha = 0$. As we recall that the Ephaptic system (3.6) decouples when $\alpha = 0$, we note that the traveling wave solution (φ_0, c_0), as well as any translates, satisfies both equations. Next, to ensure persistence of solutions when the coupling is nonzero we introduce a degree of freedom in the form of a phase delay β_0. When $\alpha = 0$ the solution to the Ephaptic system that we will continue from, now takes the form

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}(\xi) = \begin{pmatrix} \varphi_0(\xi) \\ \varphi_0(\xi+\beta_0) \end{pmatrix}. \tag{3.33}$$

The associated zero finding problem is given by the nonlinear map

$$\mathscr{G}(\phi_1, \phi_2, c, \alpha) = -c\sum_{j=1}^{3} B_j(\alpha)\phi'(\xi+r_j) - \sum_{j=1}^{3} A_j(\alpha,d)\phi(\xi+r_j) + \sum_{j=1}^{3} B_j(\alpha) f(\phi(\xi+r_j)),$$

and using (3.14) the linearization about $(\varphi_1, \varphi_2, c_0)$ is

$$D_{1,2,3}\mathscr{G}(\varphi_1, \varphi_2, c_0, 0)(\psi_1 \psi_2, b) = \begin{pmatrix} \Lambda_{c_0,\varphi_1} & 0 & -\varphi_1' \\ 0 & \Lambda_{c_0,\varphi_2} & -\varphi_2' \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ b \end{pmatrix}.$$

Next, we translate the known solution $(\varphi_1, \varphi_2, c_0)$ for $\alpha = 0$ to the origin by defining the new map $G: W_0^{1,2} \times W^{1,2} \times \mathbb{R} \to L^2 \times L^2$ as

$$G(\psi_1, \psi_2, b; \alpha) = \mathscr{G}(\varphi_1 + \psi_1, \varphi_2 + \psi_2, c_0 + b; \alpha). \tag{3.34}$$

By making the identification $x = (\psi_1, \psi_2, b)$ the linear map $D_1G(0;0)$ takes the form

$$D_1G(0;0)x = D_{1,2,3}\mathscr{G}(\varphi_1, \varphi_2, c_0, 0)x =: Ax. \tag{3.35}$$

Separating the nonlinear map G into linear and nonlinear parts, we write

$$G(x;\alpha) = Ax - N(x;\alpha), \tag{3.36}$$

where

$$N(x;\alpha) = Ax - G(x;\alpha).$$

Note, $G(0;0) = 0$ implies $N(0;0) = 0$. In addition, $D_1N(0;0) = 0$ since

$$D_1N(0;0) = A - D_1G(0;0) = 0 \tag{3.37}$$

The fact that $D_1G(0;0)$ does not yield an isomorphism precludes the direct application of the implicit function theorem. To overcome the problematic $D_1(0;0)$ the Lyapunov-Schmidt method will be employed. We define the following shorthands: $X = W_0^{1,2} \times W^{1,2} \times \mathbb{R}$, and $Z = L^2 \times L^2$. Paralleling the setup of the Lyapunov-Schmidt method in [15] we define the following projections

$$U : X \to X, \qquad X_U = UX = \mathscr{N}(A), \tag{3.38}$$

$$E : Z \to Z, \qquad Z_E = EZ = \mathscr{R}(A). \tag{3.39}$$

Note, the complementary spaces are $X_{I-U} = (I-U)X = \mathscr{N}(A)^{\perp}$ and $Z_{I-E} = (I-E)Z = \mathscr{R}(A)^{\perp}$. The linear operator A has a right inverse $K : Z_E \to X_{I-U}$

$$
\begin{aligned}
AK &= I, && \text{on } Z_E, \\
KA &= I-U, && \text{on } X.
\end{aligned}
$$

In the sequel, elements $x \in X$ are decomposed into elements of $\mathscr{N}(A)$ and into elements of the complement of the null space $\mathscr{N}(A)^{\perp}$ as

$$
x = y + z, \qquad \text{where } y \in X_u \text{ and } z \in X_{I-U}. \tag{3.40}
$$

With this setup, the Lyapunov-Schmidt equations are

$$
z - (KE)N(y+z;\alpha) = 0 \tag{3.41}
$$

$$
(I-E)N(y+z;\alpha) = 0. \tag{3.42}
$$

Before continuing it is worthwhile to prescribe the form of y:

$$
y \in X_u \implies y = \gamma \begin{pmatrix} 0 \\ \varphi_2' \\ 0 \end{pmatrix} =: \gamma y_0, \text{ for } \gamma \in \mathbb{R}. \tag{3.43}
$$

In addition, z takes the form

$$
z \in X_{I-U} \implies z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \text{ with } \int_{-\infty}^{\infty} \phi_2' z_2 d\xi = 0. \tag{3.44}
$$

An application of the implicit function theorem to eq. (3.41) gives the existence of the solution $z = z(\gamma,\alpha)$, with $z(0,0) = 0$. The vector in orthogonal complement of the range of A we call w

$$w(\xi) := \begin{pmatrix} w_1(\xi) \\ w_2(\xi) \end{pmatrix} = \begin{pmatrix} w_0(\xi) \\ -w_0(\xi+\beta_0) \end{pmatrix}. \tag{3.45}$$

The bifurcation equation (3.42) can be rewritten as

$$\hat{g}(\gamma,\alpha) = (I-E)N(\gamma y_0 + z(\gamma y_0,\alpha);\alpha) \tag{3.46}$$

$$= < \frac{w}{||w||}, N(\gamma y_0 + z(\gamma y_0,\alpha);\alpha) >= 0. \tag{3.47}$$

Note, $\hat{g}(0,0) = 0$ since $N(0;0) = 0$. Let g be the rescaling of $\hat{g}$

$$g(\gamma,\alpha) = ||w||\hat{g}(\gamma,\alpha). \tag{3.48}$$

The determination of the type of bifurcation g undergoes will require the necessary computation of the partial derivatives with respect to γ and α. The partial derivative with respect to γ is

$$g_\gamma(0,0) = < w, D_1N(0;0)x_\gamma(0;0) >= 0, \tag{3.49}$$

since $D_1N(0;0) = 0$ from (3.37). This result is expected, for if $g_\gamma(0;0) \neq 0$ then the implicit function theorem could have been directly applied to (3.34).

3.1.1 Saddle-Node Bifurcation

The Ephaptic system undergoes a bifurcation when $\alpha = 0$. Through the computation of the partial derivatives we will demonstrate that the Ephaptic system satisfies the conditions under which a saddle-node bifurcation occurs, assuming the general notation $g(\gamma_0, \alpha_0) = 0$:

$$D_\gamma g(\gamma_0, \alpha_0) = 0, \quad D_\alpha g(\gamma_0, \alpha_0) \neq 0, \quad D_{\gamma\gamma} g(\gamma_0, \alpha_0) \neq 0. \tag{3.50}$$

If g is expanded in terms of the powers of γ

$$g(\gamma, \alpha) = a_0(\alpha) + a_1(\alpha)\gamma + a_2(\alpha)\gamma^2 + \dots \tag{3.51}$$

then normal form for the saddle-node bifurcation will take

$$g(\gamma, \alpha) = \alpha - \gamma^2. \tag{3.52}$$

3.1.2 Computation of the Derivatives

To facilitate the rewriting of (3.47) we begin by manipulating N

$$\begin{aligned} N(\gamma y_0 + z(\gamma y_0, \alpha); \alpha) &= A(\gamma y_0 + z(\gamma y_0, \alpha)) - G(\gamma y_0 + z(\gamma y_0, \alpha); \alpha) \\ &= Az(\gamma y_0, \alpha) - G(\gamma y_0 + z(\gamma y_0, \alpha); \alpha), \end{aligned} \tag{3.53}$$

since $y_0 \in \mathcal{N}(A)$. Continuing with simplification in mind, we also expand the expression of G

$$G(\gamma y_0 + z(\gamma y_0;\alpha);\alpha) = \mathcal{G}(\gamma \begin{pmatrix} 0 \\ \varphi_2' \\ 0 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ b \end{pmatrix} + \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ c_0 \end{pmatrix};\alpha) \tag{3.54}$$

$$= \sum_{j=1}^{3} B_j(\alpha)[-(c_0+b)\phi'(\xi+r_j) + f(\phi(\xi+r_j))] - A_j(\alpha)\phi(\xi+r_j), \tag{3.55}$$

where ϕ is

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \gamma \begin{pmatrix} 0 \\ \varphi_2' \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{3.56}$$

Relating the original perturbative terms ψ_1, ψ_2 to (3.56) we have

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \gamma \begin{pmatrix} 0 \\ \varphi_2' \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.57}$$

Recall, the coefficients of the Ephaptic system are given by (3.7) and (3.8). Isolating the contribution of the parameter A, we introduce new $\tilde{B}$ coefficients. We write $B_1(\alpha) = I + \kappa A \tilde{B}_1(\alpha)$ where

$$\tilde{B}_1(\alpha) = \begin{pmatrix} \alpha^2 & -\alpha \\ -\alpha & \alpha^2 \end{pmatrix}, \tag{3.58}$$

and for $j = 2,3$ let $\tilde{B}_j(\alpha) = \frac{1}{\kappa A} B_j(\alpha)$

$$\begin{array}{cc} \tilde{B}_2(\alpha) & \tilde{B}_3(\alpha) \\ \begin{pmatrix} 0 & \alpha \\ 0 & -\alpha^2 \end{pmatrix}, & \begin{pmatrix} -\alpha^2 & 0 \\ \alpha & 0 \end{pmatrix}. \end{array} \tag{3.59}$$

Then, (3.55) can be rewritten as the sum of an expression with no dependence on the parameter A and an expression that does depend

$$\begin{aligned} G(\gamma y_0 + z(\gamma;\alpha);\alpha) = &\left[-(c_0+b)\phi_\xi' + f(\phi_\xi) - \sum_{j=1}^{3} A_j(\alpha)\phi_\xi(r_j) \right] \\ &+ \kappa A \sum_{j=1}^{3} \tilde{B}_j(\alpha)[-(c_0+b)\phi_\xi'(r_j) + f(\phi_\xi(r_j))] \end{aligned} \tag{3.60}$$

$$\begin{aligned} = &\left[-(c_0+b)\phi_\xi' + f(\phi_\xi) - \kappa d \begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 \end{pmatrix} \Delta\phi_\xi \right] \\ &+ \kappa A \begin{pmatrix} -\alpha^2\delta_+ & -\alpha\delta_- \\ \alpha\delta_+ & \alpha^2\delta_- \end{pmatrix} [-(c_0+b)\phi_\xi' + f(\phi_\xi)], \end{aligned} \tag{3.61}$$

where the second, forward, and backward differences are defined as

$$\begin{aligned} \Delta h(\xi) &= h(\xi+1) - 2h(\xi) + h(\xi-1), \\ \delta_+ h(\xi) &= h(\xi+1) - h(\xi), \\ \delta_- h(\xi) &= h(\xi) - h(\xi-1). \end{aligned} \tag{3.62}$$

The shorthand $f(\varphi)$ with $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ refers to component-wise application of f

$$f(\varphi) = \begin{pmatrix} f(\varphi_1) \\ f(\varphi_2) \end{pmatrix}.$$

Expanding f about φ we have

$$f(\varphi+\psi) = f(\varphi) + D_\varphi f(\varphi)\psi + \frac{D_\varphi^2 f(\varphi)}{2!}(\psi,\psi) + O(\psi^3) \tag{3.63}$$

where

$$D_\varphi^j f(\varphi)(\psi,\dots) = \begin{pmatrix} f^{(j)}(\varphi_1)\psi_1^j \\ f^{(j)}(\varphi_2)\psi_2^j \end{pmatrix}. \tag{3.64}$$

The expansion about φ will be used in N to cancel the linear terms in ψ.

By rewriting ϕ as $\psi+\varphi$, setting $A=0$ in (3.61), and expanding $f(\varphi+\psi)$, the terms in G can be regrouped to prepare for cancellation

$$\begin{aligned}
&G(\gamma y_0 + z(\gamma;\alpha);\alpha)\mid_{A=0} \\
&\qquad = -(c_0+b)(\psi+\varphi)' + f(\psi+\varphi) - \kappa d \begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 \end{pmatrix}(\Delta\psi + \Delta\varphi) \\
&\qquad = (-c_0\varphi' + f(\varphi)) + \left[(-c_0\psi' + D_\varphi f(\varphi)\psi) - b\varphi'\right] - b\psi' + \frac{D_\varphi^2 f(\varphi)}{2!}(\psi,\psi) \\
&\qquad\quad O(\psi^3) - \kappa d \begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 \end{pmatrix}(\Delta\psi + \Delta\varphi).
\end{aligned} \tag{3.65}$$

By rewriting $\kappa = 1 + \frac{\alpha^2}{1-\alpha^2}$ we have the decomposition

$$\kappa \begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \kappa \begin{pmatrix} \alpha^2 & -\alpha \\ -\alpha & \alpha^2 \end{pmatrix}. \tag{3.66}$$

The subscript 'Lin' in the form of A_{Lin} is used to distinguish the fact that we are referring to the linear operator named A in Lyapunov-Schmidt (3.35), and not the alignment parameter A. We use that fact that (φ, c_0) satisfies the nonlinear differential equation, and the definition of A_{Lin} to rewrite (3.65)

$$\begin{aligned}
&G(\gamma y_0 + z(\gamma;\alpha);\alpha) \mid_{A=0} \\
&= [-c_0\varphi' - d\Delta\varphi + f(\varphi)] + \left[-c_0\psi' - d\Delta\psi + D_\varphi f(\varphi)\psi - b\varphi'\right] - b\psi' \\
&\qquad + \frac{D^2_\varphi f(\varphi)}{2!}(\psi,\psi) + O(\psi^3) + \kappa d \begin{pmatrix} -\alpha^2 & \alpha \\ \alpha & -\alpha^2 \end{pmatrix} (\Delta\psi + \Delta\varphi) \\
&= A_{Lin} \begin{pmatrix} \psi \\ b \end{pmatrix} - b\psi' + \frac{D^2_\varphi f(\varphi)}{2!}(\psi,\psi) + O(\psi^3) + \kappa d \begin{pmatrix} -\alpha^2 & \alpha \\ \alpha & -\alpha^2 \end{pmatrix} (\Delta\psi + \Delta\varphi).
\end{aligned}$$

When $A = 0$ it follows that the nonlinear term N is

$$\begin{aligned}
&N(\psi, b;\alpha) \mid_{A=0} = \\
&-\left[-b\psi' + \frac{D^2_\varphi f(\varphi)}{2!}(\psi,\psi) + O(\psi^3) + \kappa d \begin{pmatrix} -\alpha^2 & \alpha \\ \alpha & -\alpha^2 \end{pmatrix} (\Delta\psi + \Delta\varphi)\right].
\end{aligned} \tag{3.67}$$

When $A \neq 0$ an additional term $H = N - N_{A=0} = -(G - G|_{A=0})$ from (3.61) is included:

$$H(\psi, b; \alpha) = -\kappa A \begin{pmatrix} -\alpha^2\delta_+ & -\alpha\delta_- \\ \alpha\delta_+ & \alpha^2\delta_- \end{pmatrix} [-(c_0 + b)(\varphi + \psi)' + f(\varphi + \psi)]. \tag{3.68}$$

3.1.3 Useful Computations

We find the partial derivative of N with respect to α is:

$$\begin{aligned} &D_3N(\psi, b; \alpha) = \\ &\qquad -d\left[\frac{1}{1-\alpha^2}\begin{pmatrix} -2\alpha & 1 \\ 1 & -2\alpha \end{pmatrix} + \frac{2\alpha}{(1-\alpha^2)^2}\begin{pmatrix} -\alpha^2 & \alpha \\ \alpha & -\alpha^2 \end{pmatrix}\right](\Delta\psi + \Delta\varphi), \\ &D_3N(0,0;0) = -d\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Delta\varphi = -d\begin{pmatrix} \Delta\varphi_2 \\ \Delta\varphi_1 \end{pmatrix}. \end{aligned} \tag{3.69}$$

The partial derivative of H with respect to α is:

$$\begin{aligned} D_3H(\psi, b; \alpha) = -A &\left[\frac{1}{1-\alpha^2}\begin{pmatrix} -2\alpha\delta_+ & -\delta_- \\ \delta_+ & 2\alpha\delta_- \end{pmatrix}\right. \\ &\left. + \frac{2\alpha}{(1-\alpha^2)^2}\begin{pmatrix} -\alpha^2\delta_+ & -\alpha\delta_- \\ \alpha\delta_+ & \alpha^2\delta_- \end{pmatrix}\right][-(c_0 + b)(\varphi + \psi)' + f(\varphi + \psi)]. \end{aligned}$$

Using the fact that (φ, c_0) is a solution to the differential equation when $\alpha = 0$, we have $d\Delta\varphi = -c_0\varphi' + f(\varphi)$, and

$$D_3H(0,0;0) = -Ad\begin{pmatrix} 0 & -\delta_- \\ \delta_+ & 0 \end{pmatrix}\Delta\varphi = -Ad\begin{pmatrix} -\delta_-\Delta\varphi_2 \\ \delta_+\Delta\varphi_1 \end{pmatrix}. \tag{3.70}$$

To help compute $g_{\gamma,\gamma}$ we write out the derivatives of $D_1N(x;0)$ and $D_1^2N(x;0)$. We interchange among the forms $x = (\psi_1, \psi_2, b)^T$, $\psi = (\psi_1, \psi_2)^T$, and $v = (v_1, v_2, v_3)^T$

$$D_1N(x;0)v = b\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} + v_3\psi' - \begin{pmatrix} f''(\varphi_1)\psi_1 v_1 \\ f''(\varphi_2)\psi_2 v_2 \end{pmatrix} + O(\psi^2), \tag{3.71}$$

and

$$D_1^2N(x;0)(v,v) = v_3\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} + v_3\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} - \begin{pmatrix} f''(\varphi_1)v_1^2 \\ f''(\varphi_2)v_2^2 \end{pmatrix} + O(\psi). \tag{3.72}$$

Evaluating the second derivative at $x = (\psi_1, \psi_2, b) = 0$ gives

$$D_1^2N(0;0)(v,v) = 2v_3\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} - \begin{pmatrix} f''(\varphi_1)v_1^2 \\ f''(\varphi_2)v_2^2 \end{pmatrix}. \tag{3.73}$$

3.1.4 $g_\alpha(0,0)$

Our goal in examining $g_\alpha(0,0)$ is to gather information about $a_0(\alpha)$ in (3.51). We first calculate the value of g_a when A is fixed at 0. We note that $\frac{\partial}{\partial\alpha}\gamma y_0 = 0$ which implies $x_\alpha = z_\alpha$. The partial derivative of g with respect to α is now

$$g_\alpha(\gamma,\alpha)|_{A=0} = < w, D_1N(x;\alpha)z_\alpha + D_2N(x;\alpha) > \tag{3.74}$$

When evaluating at $(x,\alpha)=(0,0)$ the term $D_1N(0;0)z_\alpha$ is 0 from (3.37), and by using (3.69) to simplify, we have

$$
\begin{aligned}
g_\alpha(0,0)|_{A=0} &=< w, D_2N(0;0) > \\
&=< w, -d\begin{pmatrix}\Delta\varphi_2 \\ \Delta\varphi_1\end{pmatrix} >= -d\int w_1\Delta\varphi_2 + w_2\Delta\varphi_1 d\xi. \qquad (3.75)
\end{aligned}
$$

Rewriting φ_1, φ_2 using (3.33) we have in the case when $A=0$

$$
\begin{aligned}
g_\alpha(0,0)|_{A=0} &= -d\int w_0(\xi)\Delta\varphi_0(\xi+\beta_0) - w_0(\xi+\beta_0)\Delta\varphi_0 d\xi \\
&= -dI_1, \qquad (3.76)
\end{aligned}
$$

where

$$
I_1 = \int w_0(\xi)[\Delta\varphi_0(\xi+\beta_0) - \Delta\varphi_0(\xi-\beta_0)]d\xi. \qquad (3.77)
$$

When $A \neq 0$ the additional contribution from H, noting that the terms in $D_1H(x;\alpha)$ all include powers of α, can be written as

$$
\begin{aligned}
I_H &=< w, D_1H(0;0)z_\alpha + D_2H(0;0) >=< w, D_2H(0;0) > \\
&= Ad\int [w_1(\delta_-\Delta\varphi_2) - w_2(\delta_+\Delta\varphi_1)]d\xi = Ad[I_1+I_2] \qquad (3.78)
\end{aligned}
$$

where

$$
\begin{aligned}
I_2 &= \int -w_2(\xi)\Delta\varphi_1(\xi+1) - w_1(\xi)\Delta\varphi_2(\xi-1)d\xi \\
&= \int w_0(\xi+\beta_0)\Delta\varphi_0(\xi+1) - w_0(\xi)\Delta\varphi_0(\xi+\beta_0-1)d\xi \\
&= \int w_0(\xi)[\Delta\varphi_0(\xi+(1-\beta_0)) - \Delta\varphi_0(\xi-(1-\beta_0))]d\xi. \qquad (3.79)
\end{aligned}
$$

The overall value of $g_a(0,0)$ can be written as

$$
\begin{aligned}
g_\alpha(0,0) &= -dI_1 + I_H, \\
&= -d[(1-A)I_1 - AI_2]. \qquad (3.80)
\end{aligned}
$$

3.1.5 $g_{\gamma\gamma}(0,0)$

The calculation of the second partial derivative with respect to γ is given by

$$
g_{\gamma\gamma}(\gamma,\alpha) = < w, D_1N(x;\alpha)x_{\gamma\gamma} + D_1^2N(x;\alpha)(x_\gamma, x_\gamma) > . \qquad (3.81)
$$

We will rely on the fact that $x_\gamma(0;0) = y_0$, and this fact is a simple consequence of differentiating both sides of (3.41) with respect to γ and evaluating at $(0,0)$

$$
z_\gamma(0;0) - (KE)D_1N(0;0)(y_0 + z_\gamma(0;0)) = 0, \qquad (3.82)
$$

and noting the fact that $D_1N(0;0) = 0$. Utilizing the form of $D_1^2N(0;0)$ in (3.73), noting that the third component of $y_0^T = (0, \phi_2', 0)^T$ from (3.43) is zero, and using $x_\gamma(0;0) = y_0$ gives the following reduction

$$
\begin{aligned}
g_{\gamma\gamma}(0,0) &= < w, D_1^2N(0;0)(y_0,y_0) > = < w, \begin{pmatrix} 0 \\ -f''(\varphi_2)(\varphi_2')^2 \end{pmatrix} > \\
&= \int w_0 f''(\varphi_0)(\varphi_0')^2 d\xi. \qquad (3.83)
\end{aligned}
$$

There is no additional contribution to $g_{\gamma\gamma}$ when $A \neq 0$ due to the presence of powers of α in all terms of H in (3.68).

3.1.6 Numerical Computation of Partial Derivatives

This section discusses the computation of the required partial derivatives for the Lyapunov-Schmidt method. Even if we are fortunate enough to obtain a closed form solution as in [1] or [37] for their respective nonlinearities, the function w in the kernel of the adjoint is not necessarily known. We employs numerics to compute the function w and calculate the required integrals. Recall, our calculations are performed for the cubic-like nonlinearity (2.118) considered in [1]

$$f(u) = h(u)u(u-1), \text{ where } h(u) = \left(\frac{d\gamma(2u-1)}{\gamma[1-u]u+1} - cb\right).$$

With the cubic-like nonlinearity we note that a closed form solution exists to (3.12), and is

$$\varphi(\xi) = \frac{1}{2}\left[1+\tanh\left(\frac{b}{2}\xi\right)\right]. \tag{3.84}$$

The linearization of f is made about (3.84), and the computation of $g_{\gamma\gamma}(0;0)$, which relies on f'', is given from (2.122), and is

$$f''(u) = 2h(u) + 2(2u-1)h'(u) + u(u-1)h''(u).$$

3.1.7 Bifurcation in the Ephaptic System

By examining Figures 3.3 and 3.4 we can see the conditions under which $g_\alpha(0,0) > 0$ and $g_{\gamma\gamma}(0,0) < 0$. In Figure 3.3 if β and a are chosen from either the northwest or southeast regions the value of g_α is positive. In Figure 3.4 the value of $g_{\gamma\gamma}$ is negative, and this fact holds regardless of the value of the parameters β or A.

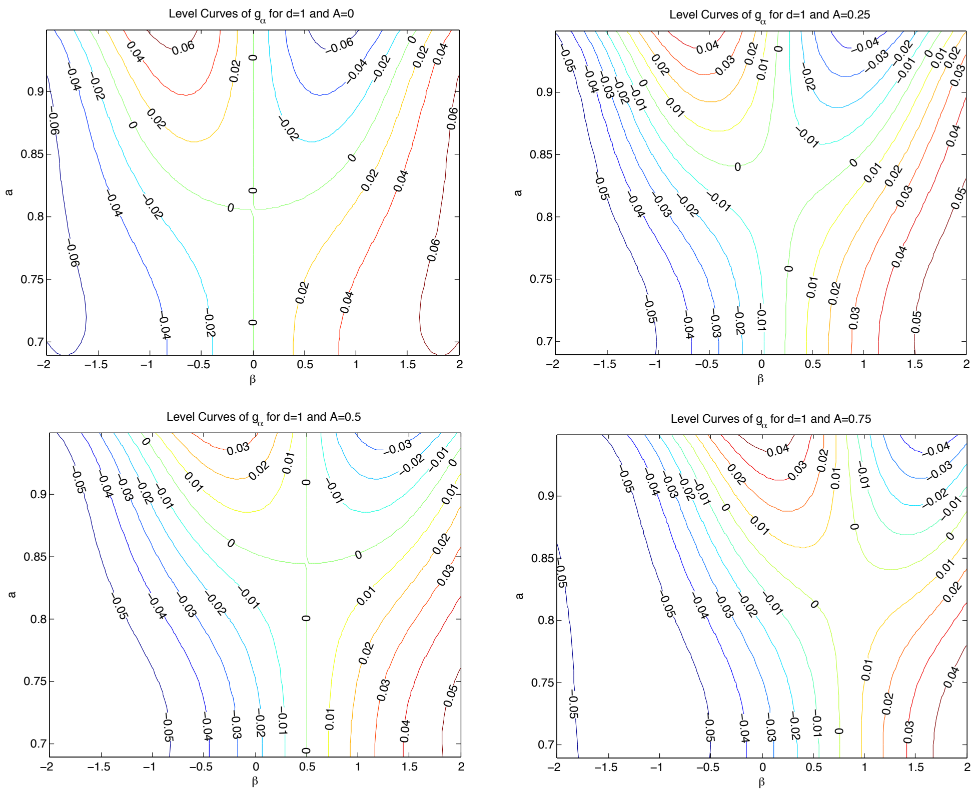


Figure 3.3: Zero level curves of g_α for $A = 0, 0.25, 0.5, 0.75$.

Figure 3.5 depicts and emphasizes the change in orientation of the zero level curves of g_α. As the parameter A is varied the nature of the set of positive values of g_α varies. When $A \in [0, .5]$ the set $\{(\beta, a) | g_\alpha > 0\}$ is disconnected, and connected when $A \in (-.5, 0) \cup (.5, 1)$.

Hence the condition for a saddle-node bifurcation (3.50) is met as long as g_α and $g_{\gamma\gamma}$ take opposite signs with the understanding that the diffusion coefficient d is large enough to give a wave speed $c > 0$:

Proposition 3.1.1. *For $d > d^*$ and appropriately chosen (β, a), then the Ephaptic system (3.6) undergoes a saddle-node bifurcation at $\alpha = 0$.*

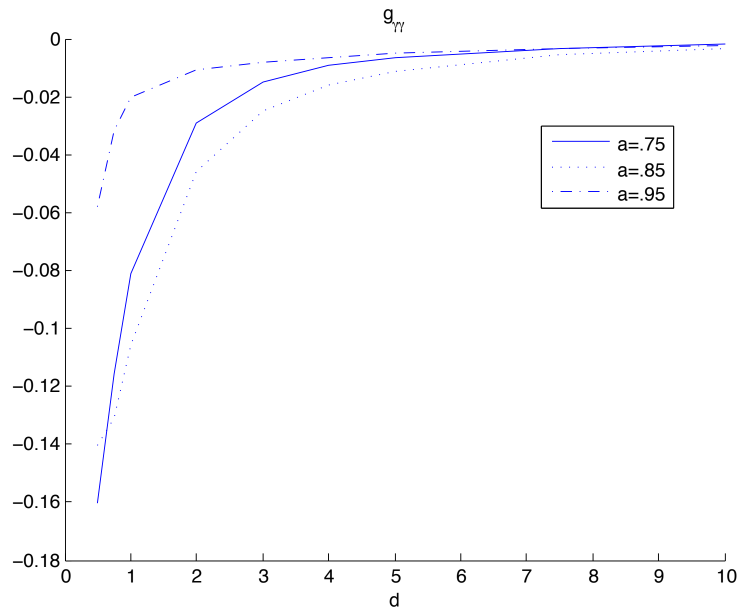


Figure 3.4: The sign of $g_{\gamma\gamma}$ does not vary.

3.2 Conclusion

Neutral equations of mixed type are similar to neutral delay equations, but contain both advances and delays and are not well posed as IVP. The linear Fredholm theory builds on the linear theory of Mallet-Paret for mixed type problems by utilizing exponential dichotomies for the difference operator on the neutral side. The Lyapunov-Schmidt method is used to detect a saddle-node bifurcation in the Ephaptic coupled nerve fiber problem.

Presently, the Green's function construction detailed in section 2.3 relies upon shifts being rationally related. The models considered in the example section happen to be of this type. The case of shifts not being rationally related is an area of future work, and is a necessary component for traveling plane waves. In addition, the question of stability for the branches of the saddle-node bifurcation remains for future work.

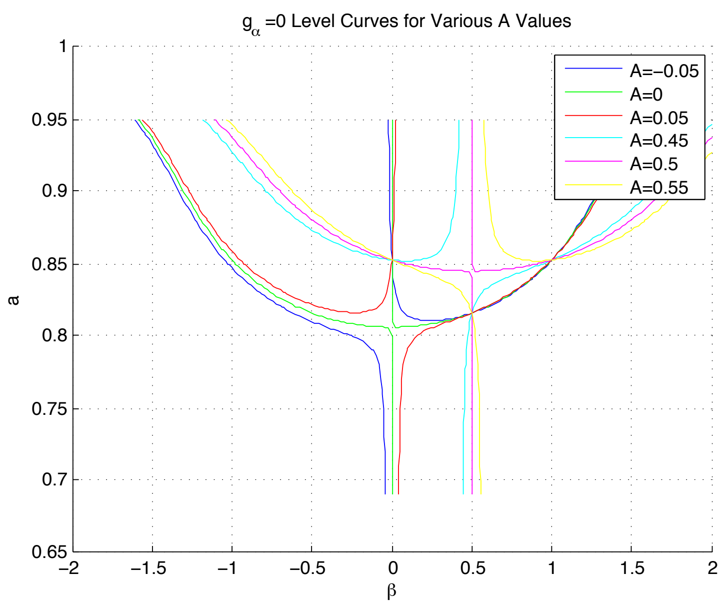


Figure 3.5: The orientation of the zero level curves changes orientation as A crosses either 0 and 0.5.

Chapter 4

Numerical Studies of Gap-Ephaptic System

In [12], Bose studied a parallel nerve fiber model wherein each nerve fiber is described by the FitzHugh-Nagumo equations in its uncoupled state. Of particular interest in our studies is the form of coupling considered. Save a change in notation, the model considered by Bose and earlier by Keener [28] is

$$\begin{aligned}\varepsilon u_{1t} &= \varepsilon^2 u_{1xx} + f(u_1) - v_1 + \nu(u_2 - u_1),\\ \varepsilon u_{2t} &= \varepsilon^2 u_{2xx} + f(u_2) - v_2 + \nu(u_1 - u_2),\\ v_{1t} &= u_1 - \gamma v_1,\\ v_{2t} &= u_2 - \gamma v_2.\end{aligned} \tag{4.1}$$

In the context of [12], the coupling parameter ν has the significance of being excitatory or inhibitory depending upon the sign of ν. In the context of our model, presented Section 4.1, this form of coupling has differing behavior depending on the sign of parameter. In particular, this second form of coupling forces the solution to lose monotonicity and increases the wave speed when ν takes a negative sign. This wave speed increase is in contrast to Ephaptic coupling introduced in Chapter 3.

Our numerical computations utilize the `chebfun` package, a collection of Chebyshev tools, as detailed in [7]. The benefit with utilizing a spectral method for solving functional differential equations is the ability to evaluate shifts which may not fall upon a node already present in the mesh. Via the global representation of the solution, shifts and arbitrary translations are able to be accommodated.

4.1 Gap-Ephaptic Model Equations

The model under consideration is an adaptation of the Ephaptic model considered in [11],[5],[29] with the added coupling in ν considered by Bose in (4.1). This second form of coupling inspired by Bose, we term the gap coupling. This terminology is in reference to the physiology of nerves and potential coupling between gap-junctions. The model is of the following form:

$$B_{-1}W_{n-1} + B_0 W_n + B_1 W_{n+1} = A_{-1}V_{n-1} + A_0 V_n + A_1 V_{n+1} + CV_n, \tag{4.2}$$

where $W_n = \dot{V}_n + f(V_n)$ and $V_n = [V_n^{(1)} \quad V_n^{(2)}]^T$. The B_j and A_j coefficients depend on the parameters α, d, A; the matrix coefficient C depends on the parameter ν. The B and A coefficients are the same as considered in (3.7) (3.8) and, lastly, the matrix coefficient C is of the form

$$C = \nu \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.3}$$

By imposing the traveling wave ansatz $v_n = \phi(n - ct)$ with $\xi = n - ct$ the traveling wave equations become

$$-c \sum_{j=-1}^{1} B_j \phi'(\xi + r_j) = \sum_{j=-1}^{1} \Big[A_j \phi(\xi + r_j) - B_j f(\phi(\xi + r_j)) \Big] + C\phi(\xi), \tag{4.4}$$

where $r_{-1} = -1, r_0 = 0$, and $r_1 = +1$.

4.2 Nonlinear Equations

In the case that the coupling parameters take the values $(\alpha, \nu) = (0,0)$, the equations decouple into two identical copies of the discrete Nagumo equation (3.12)

The more general model in (4.3) is amenable to a similar setup in [29]. Denote the solution to (3.12) by (φ_0, c_0). When the coupling is off, i.e. $(\alpha, \nu) = (0,0)$, we note that the traveling wave solution (φ_0, c_0), as well as any translates, satisfy each uncoupled scalar equation. In a similar manner to [29] we introduce a phase delay via the variable β to ensure persistence of solutions for nonzero coupling. Our candidate solution to the system (4.3), when $(\alpha, \nu) = (0,0)$, takes the form

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}(\xi) = \begin{pmatrix} \varphi_0(\xi) \\ \varphi_0(\xi + \beta_0) \end{pmatrix}. \tag{4.5}$$

The associated zero finding problem is given by the nonlinear map

$$\begin{aligned} \mathscr{G}(\phi_1, \phi_2, c; \alpha, \nu) = -c \sum_{j=-1}^{1} B_j(\alpha)\phi'(\xi + r_j) - \sum_{j=-1}^{1} A_j(\alpha, d)\phi(\xi + r_j) \\ + \sum_{j=-1}^{1} B_j(\alpha) f(\phi(\xi + r_j)) + C(\nu)\phi(\xi), \end{aligned}$$

and using

$$D_{1,2,3}\mathscr{G}(\varphi_1, \varphi_2, c_0; 0, 0)(\psi_1, \psi_2, b) = \begin{pmatrix} \Lambda_{c_0,\varphi_1} & 0 & -\varphi_1' \\ 0 & \Lambda_{c_0,\varphi_2} & -\varphi_2' \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ b \end{pmatrix}.$$

Translating the solution $(\varphi_1, \varphi_2, c_0)$ for $(\alpha, \nu) = (0,0)$ to the origin by defining the new map $G : W_0^{1,2} \times W^{1,2} \times \mathbb{R} \to L^2 \times L^2$ as

$$G(\psi_1, \psi_2, b; \alpha, \nu) = \mathscr{G}(\varphi_1 + \psi_1, \varphi_2 + \psi_2, c_0 + b; \alpha, \nu). \tag{4.6}$$

By making the identifications $x = (\psi_1, \psi_2, b)$ and $\lambda = (\alpha, \nu)$ the linear map $D_1 G(0;0)$ takes the form

$$D_1 G(0;0)x = D_{1,2,3}\mathscr{G}(\varphi_1, \varphi_2, c_0; 0)x =: Ax. \tag{4.7}$$

4.3 Pseudo-Spectral Method

Eilbeck [17] and Eilbeck and Flesch [18] have employed a pseudo-spectral approach to solve pdes and mixed type problems, respectively. This section discusses the implementation of a pseudo-spectral method for systems of neutral equations of mixed type. In a pseudo-spectral method, a solution of two independent variables, such as $u(x,t)$, has an approximate solution $\tilde{u}$ given by the projection onto a truncated set of orthogonal basis functions $\{\psi_k(x)\}$

$$u(x,t) \approx \tilde{u}(x,t) = \sum_{k=0}^{M} c_k(t)\psi_k(x) = \Psi^T \mathbf{c}, \tag{4.8}$$

where the coefficients $\mathbf{c}(t) = [c_0(t), \ldots, c_M(t)]^T$ are determined by collocation on a mesh $\{x_i\}$ of $M+1$ points, and the column vector of the basis functions is given by $\Psi^T = [\psi_0(x), \ldots, \psi_M(x)]$.

In traveling wave problems, the solution is time independent after imposing the traveling wave ansatz $\xi = n - ct$. The wave speed c is an additional unknown that requires an additional equation to fix the translate: a phase condition, $\sigma(u(\xi)) = 0$, serves this purpose as seen below. In the time independent case, we consider approximate solutions in the following form:

$$u(\xi) \approx \tilde{u}(\xi) = \sum_{k=0}^{M} c_k \psi_k(\xi). \tag{4.9}$$

Ideally the basis functions, $\{\psi_k(x)\}$, are also eigenfunctions of the linear operator

$$(\mathscr{L}\psi_k)(\xi) = \lambda_k \psi_k(\xi), \tag{4.10}$$

but when this is not the case we may be able to utilize recurrence relationships, see Sec. 4.4.1, to rewrite the action of $\mathscr{L}$ in terms of the basis

$$\mathscr{L}\left[\sum_{k=0}^{M} c_k \psi_k(\xi)\right] = \sum_{k=0}^{M} b_k \psi_k(\xi). \tag{4.11}$$

We are interested in traveling wave problems posed on the real line, but for numerics, the problem is truncated to the finite interval $\mathscr{I} = [-L, L]$. Let $\mathscr{L}$ denote the linear operator on the interval $\mathscr{I}$ with the following inhomogeneous boundary conditions (BC):

$$u(-L) = 0 \text{ and } u(L) = 1. \tag{BC}$$

In the context of traveling waves, we must also include another condition to match the number of unknowns. As detailed in [16], two ways of fixing the translate are by imposing a local standard phase condition $\sigma_1(u) = 0$ or enforcing a global integral condition $\sigma_2(u) = 0$

$$\sigma_1(u) = u(0) - \frac{1}{2}, \tag{SPC}$$

$$\sigma_2(u) = \int_{-L}^{L} (u(x) - u^0(x))^T (u^0)'(x) dx, \tag{IPC}$$

where $u^0(x)$ can be viewed as the initial iterate while applying Newton's method.

4.3.1 Nonlinear Equations

In the scalar traveling wave equation

$$-cu' = \Delta_1 u - f(u) \tag{4.12}$$

with Δ_1 denoting the discrete Laplacian, the substitution of the approximate solution (4.9) yields

$$-c \sum_{k=0}^{M} a_k \psi_k(\xi) = \sum_{k=0}^{M} b_k \psi_k(\xi) - f(\tilde{u}) + r(\xi), \tag{4.13}$$

for some a_k and b_k depending on the action of $d/d\xi$ and Δ_1 on $\{\psi_k\}$ and a residual function $r(\xi)$. In the pure spectral method the residual function is forced to be orthogonal to the $M+1$ basis functions, however a collocation approach is employed in the pseudo-spectral method. Instead, the residual $r(x_i, t) = 0$ is forced to zero at the set of mesh points $\{x_i\}_{i=0}^{M}$

$$r(\xi_i) = 0 = -c \sum_{k=0}^{M} a_k \psi_k(\xi_i) - \sum_{k=0}^{M} b_k \psi_k(\xi_i) + f(\tilde{u}(\xi_i)). \tag{4.14}$$

To facilitate a compact notation we let $P = \{p_{ik}\} = \{\psi_k(x_i)\}$ with $\tilde{u} = [\tilde{u}(\xi_0), \ldots, \tilde{u}(\xi_M)]^T$ giving

$$\tilde{u} = P\mathscr{C}. \tag{4.15}$$

If the coefficients a_k and b_k given by the vectors $\mathbf{a}$ and $\mathbf{b}$ are related to $\mathbf{c}$ by the matrices D_1 and D_2 as

$$\mathbf{a} = D_1\mathbf{c}, \quad \mathbf{b} = D_2\mathbf{c}, \tag{4.16}$$

then the system becomes

$$-cPD_1\mathbf{c} - PD_2\mathbf{c} - f(P\mathbf{c}) = 0. \tag{4.17}$$

Newton's method is then equivalent to solving the following nonlinear system

$$G(\mathbf{c}, c) = -cPD_1\mathbf{c} - PD_2\mathbf{c} - f(P\mathbf{c}) = 0. \tag{4.18}$$

4.4 Chebyshev Series

When seeking traveling wave front solutions with inhomogeneous boundary conditions of the form (BC), a Chebyshev basis is more appropriate than a Fourier basis due to the lack of periodicity. A truncated Chebyshev series takes the form

$$p(x) = \sum_{k=0}^{M} a_k T_k(x), \quad x \in [-1, 1]. \tag{4.19}$$

The basis functions $T_k(x)$ are called Chebyshev polynomials of the first kind. The basic recurrence relationship among the Chebyshev polynomials is

$$T_{k+1} = 2xT_k(x) - T_{k-1}(x), \quad \text{for } k \geq 1, \tag{4.20}$$

with $T_0(x) = 1$ and $T_1(x) = x$. Other Chebyshev properties are detailed in [14] and [42]. The non-uniformly spaced Chebyshev points of the second kind $\{x_i\}$ are related to a uniform discretization of the unit circle. Let the top-half of the unit circle be uniformly discretized by $M+1$ points,

$$\theta_i = \frac{i\pi}{M}, \quad \text{for } 0 \le i \le M, \tag{4.21}$$

then the $M+1$ Chebyshev points of the second kind $\{x_i\}$ related by the following statement:

$$x_i = \cos(\theta_i), \quad \text{for } 0 \le i \le M. \tag{4.22}$$

By rescaling the y, the independent variable ranging over the truncated real line, a Chebyshev expansion may be used:

$$x = y/L, \quad \text{for} \quad y \in \mathscr{I} = [-L, L]. \tag{4.23}$$

4.4.1 Action on Chebyshev Polynomials

In the spatially discrete problems we consider, two linear operators are utilized: the differential operator d/dx and the discrete Laplacian operator Δ_1. Whereas a Fourier basis $w_k(x) = e^{ikx}$ act as eigenfunctions of d/dx and Δ_1,

$$\begin{cases} w_k'(x) = ikw_k(x), \quad w_k(x+r) = e^{ikr}w_k(x) \\ \Delta_1 w_k(x) = (e^{ik} - 2 + e^{-ik})w_k(x) \end{cases} \tag{4.24}$$

the action on the Chebyshev polynomials is less straight-forward. The action of the differential operator on a Chebyshev series expansion requires the usage of a recursive formula to rewrite as a Chebyshev series expansion. If $p(x)$ is given by (4.19), we seek a Chebyshev series expansion for $p'(x)$ in the form

$$p'(x) = \sum_{k=0}^{M-1} b_k T_k(x). \tag{4.25}$$

The coefficients of this expansion can be found by relating the expansion coefficients of $p'(x)$ with the coefficients of $p(x)$ by the recursive formula

$$b_{k-1} = b_{k+1} + 2ka_k, \qquad 2 \leq k \leq M \tag{4.26}$$

with $b_0 = b_2/2 + a_1$ and $b_M = b_{M+1} = 0$.

4.4.2 Discrete Laplacian

For notational purposes, we let the symbol $\tilde{}$ denote the rescaled variables posed on the interval $[-1,1]$. Suppose that (4.22) and (4.23) relate the variables y, x, and θ with $y \in \mathscr{I}$ implying $x \in \tilde{\mathscr{I}}$. A shift r of the variable y, i.e. $u(y+r)$, amounts to a shift $\tilde{r} = r/L$ of x. Due to the error inherent in sampling a Chebyshev series expansion outside of $[-1,1]$, we confine our attention to $y \in \mathscr{I}_1 = [-L+1, L-1]$ for the discrete Laplacian with shifts of ± 1, and use other means to define Δ_1 on the boundary intervals $[-L, -L+1]$ and $[L-1, L]$.

We seek Chebyshev series expansions for the translation $p(x+\tilde{r})$ and ultimately $\Delta_{\tilde{r}} p(x)$ in terms of coefficients of the original expansion. As we see below, this is not possible. By exploiting the relationship between Chebyshev and Fourier series by setting $a_{-k} = a_k$, we have the following equivalence:

$$p(x) = P(\theta) = \sum_{k=-M}^{M} a_k e^{ik\theta}. \tag{4.27}$$

We aim to find the correspondence of translation of x on the spatial side and translation of θ on the angular side. We find that a uniform translation of x, amounts to a nonuniform translation of θ. Consider the following trigonometric identity

$$\sin(\omega)\sin(\phi) = \frac{\cos(\omega-\phi)-\cos(\omega+\phi)}{2}. \tag{4.28}$$

With the change of variable $\theta = \omega+\phi$ the statement

$$\cos(\theta)+2\sin(\theta-\phi)\sin(\phi) = \cos(\theta-2\phi) \tag{4.29}$$

can be used to handle translations. For each $x_i = \cos(\theta_i)$, a corresponding angle shift ϕ_i exists to satisfy the equation

$$\cos(\theta_i)+\tilde{r} = \cos(\theta_i-2\phi_i) \tag{4.30}$$

for the restricted set of $x_i \in \tilde{\mathscr{I}}_1$. The phase shift ϕ depends on θ_i, and as a result the translation $p(x+\tilde{r})$ can be computed pointwise:

$$p(x_i+\tilde{r}) = P(\theta_i-2\phi_i) = \sum_{k=-M}^{M} a_k e^{ik(\theta_i-2\phi_i)}, \quad \text{for } x_i \in \tilde{\mathscr{I}}_1. \tag{4.31}$$

This sum is an example of type 2 nonuniform discrete Fourier transform; special techniques can be employed to handle this type of transform, e.g. see [22]. We choose a straight-forward approach performing an ifft for each $x_i \in \tilde{\mathscr{I}}$. This process can be viewed as performing an ifft of a rectangular matrix column-wise with entries $a_{ki} = a_k e^{-2ikt_i}$. For each $x_i \in \tilde{\mathscr{I}}_1$, an ifft operation must be performed to extract the entry corresponding to x_i at a total cost of $O(M^2\log(M))$.

We also find that the forward and backward translations of x_i do not cancel each other to simplify computations. For $x_i \in \tilde{\mathscr{I}}_1$, if (θ_i, ϕ_i) satisfies (4.30) for $\tilde{r}$, then $(\theta_{M-i}, -\phi_i)$ satisfies (4.30) for $-\tilde{r}$. Let t_i, s_i represent the angle shifts for $\tilde{r}$ and $-\tilde{r}$, respectively. For a fixed i, unless $\theta_i = 0$ the shifts are not equal $t_i \neq -s_i$. It follows that the discrete Laplacian $\Delta_{\tilde{r}} p$ also requires an ifft for each $x_i \in \tilde{\mathscr{I}}_1$:

$$\Delta_{\tilde{r}} p(x_i) = \sum_{k=-M}^{M} a_k (e^{-2ikt_i} - 2 + e^{-2iks_i}) e^{ik\theta_i}, \quad \text{for } x_i \in \tilde{\mathscr{I}}_1. \tag{4.32}$$

4.4.3 Chebyshev Interpolation

To establish a baseline of accuracy for the Chebyshev tools being utilized, we consider the test problem of approximating the function $f_\varepsilon(y) = \tanh(y/\varepsilon)$. Pertaining to the issue of accuracy, there are two sources of truncation error: the truncation of the real line to the interval $\mathscr{I} = [-L, L]$ and the truncation of the Chebyshev series. The truncation of the Chebyshev series is examined in Table 4.1 as the maximum norm error is presented as a function of the number of Chebyshev basis functions M. The interpolation error generated by the $M-1^{th}$ degree Chebyshev polynomial is tested by evaluating u posed on $[-10, 10]$ given by the Matlab command

$$\mathtt{u} = \mathtt{chebfun}(\text{`}\mathtt{tanh(x)}\text{`}, [-10, 10], \text{`}\mathtt{length}\text{`}, \mathtt{M}).$$

As a point of reference, the adaptively sized Chebyshev polynomial on $[-10, 10]$ returned by $\mathtt{u} = \mathtt{chebfun}(\text{`}\mathtt{tanh(x)}\text{`}, [-10, 10])$ is a polynomial of degree 233 giving an interpolation error on the order of machine precision.

M	8	16	32	64	128
Error	0.26763	0.11304	9.36778e-3	6.50279e-05	2.92034e-09

Table 4.1: Interpolation Error for $L = 10$.

To assess the impact of truncating the real line, the parameter L is varied. The value of M is fixed to be 64. The maximum norm error is presented in Table 4.2 as L is varied from 1 to 20. The effect of varying ε is closely related to varying L. The interpolation of $f_\varepsilon(y)$ on $[-L,L]$ induces an interpolation error equivalent to the error present from interpolating $g(w) = f_\varepsilon(\varepsilon w) = \tanh(w)$ on the interval $[-\tilde{L},\tilde{L}] = [-L/\varepsilon, L/\varepsilon]$. If L is fixed and ε is varied, the error measurements in Table 4.2 also hold if L/ε is varied between 1 and 20.

L	1	2.5	5	10	20
Error	6.6613e-16	7.7715e-16	3.7251e-09	5.9444e-05	8.1931e-03

Table 4.2: Maximum Error for $\varepsilon = 1$ and $M = 64$.

4.4.4 Nagumo Equations

A starting solution to the two-dimensional system (4.4) contains two copies of the discrete Nagumo profile with the second component profile shifted by a phase delay of β. To obtain the solution to the discrete Nagumo system we construct a homotopy linking the Nagumo pde with the discrete Nagumo. The Nagumo pde and discrete Nagumo are given as

$$u_t = u_{xx} - f(u), \tag{4.33}$$

$$u_t = \Delta_1 u - f(u), \tag{4.34}$$

respectively, where the cubic nonlinearity is given by $f(u) = u(u-a)(u-1)$. By imposing a traveling wave ansatz $u(x,t) = \phi(x-ct)$ and the change of variable $\xi = x - ct$

the equations become

$$-c\phi'(\xi) = \phi''(\xi) - f(\phi), \tag{4.35}$$

$$-c\phi'(\xi) = \Delta_1\phi(\xi) - f(\phi). \tag{4.36}$$

The closed form solution (ϕ, c) is given for the Nagumo pde by the following formula:

$$(\phi(\xi), c) = \left(\frac{1}{2}\left(1 + \tanh\left(\frac{\xi}{2d\sqrt{2}}\right)\right), \frac{d}{\sqrt{2}}(2a-1)\right), \tag{4.37}$$

see [19] for additional discussion. The homotopy is traversed and converges to the solution of the discrete Nagumo by utliizing damped Newton's method.

4.5 Pseudo-Spectral Method for Systems

In this section we seek traveling wave solutions, $(\vec{\phi}, c)$, of d-dimensional systems of equations. Each entry in the vector of profiles, $\vec{\phi}$, corresponds to a single wave speed, c. As a result, only one moving reference frame, ξ, is considered in the change of variable

$$\begin{aligned} \xi &= n - ct \\ \tau &= t. \end{aligned} \tag{4.38}$$

Within the moving reference frame, the spatially discrete system of differential-difference equations, e.g. (4.2), may be rewritten as a system of neutral equations of mixed type in the form

$$\sum_{j=1}^{N} B_j\left[-c\vec{\phi}_\xi(\xi + r_j) + \vec{\phi}_\tau(\xi + r_j)\right] = \sum_{j=1}^{N}\left[A_j\vec{\phi}(\xi + r_j) - B_j f(\vec{\phi}(\xi + r_j))\right].$$

We restrict our attention to traveling wave solutions that are independent of time, i.e. $\vec{\phi}_\tau = \vec{0}$. The simplified equations, emphasizing the dependence on the vector of parameters, α, and eliminating the explicit denotation of vector-valued function, as in "$\vec{\phi}$", read as follows

$$-c\sum_{j=1}^{N} B_j(\alpha)\phi'(\xi+r_j) = \sum_{j=1}^{N}\Big[A_j(\alpha)\phi(\xi+r_j) - B_j(\alpha)f(\phi(\xi+r_j))\Big], \tag{4.39}$$

where $' = d/d\xi$. We must handle inhomogeneous boundary conditions as in the case of traveling wave fronts. We institute this change of dependent variable by noting that if the underlying problem is in fact a solitary wave, then $w(\xi) = 0$. The rewritten form, utilizing functional notation, is

$$\begin{aligned}
-c\sum_{j=1}^{N} B_j(\alpha)v'_\xi(r_j) &= \sum_{j=1}^{N}\Big[A_j(\alpha)v_\xi(r_j) - B_j(\alpha)g(v_\xi(r_j))\Big] + h(\xi,c),\\
g(v_\xi(r_j)) &= f(w(\xi+r_j)+v(\xi+r_j)),\\
h(\xi,c) &= \sum_{j=1}^{N}\Big[cB_j(\alpha)w'(\xi+r_j) + A_j(\alpha)w(\xi+r_j)\Big].
\end{aligned} \tag{4.40}$$

We truncate the problem to the interval $\mathscr{I} = [-L, L]$ and utilize a pseduo-spectral approach.

4.6 Continuation

Due to the error in sampling a Chebyshev polynomial outside of its domain, we view the equation (4.4) as a sum of difference operators. In this form, we expect a first or second difference to approach zero near the boundary. For notational convenience, we introduce discrete Laplacian, forward difference, and backward difference:

$$\begin{aligned}(\Delta u)(\xi) &= u(\xi+1) - 2u(\xi) + u(\xi-1),\\ (\delta_+ u)(\xi) &= u(\xi+1) - u(\xi), \\ (\delta_- u)(\xi) &= u(\xi) - u(\xi-1).\end{aligned} \tag{4.41}$$

The associate zero finding problem to (4.4) can be written as

$$G(\phi_1, \phi_2, c; \alpha, \nu) = \mathbf{B}\left[-c\begin{pmatrix}\phi_1' \\ \phi_2'\end{pmatrix} + \begin{pmatrix} f(\phi_1) \\ f(\phi_2)\end{pmatrix}\right] - \mathbf{A}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}, \tag{4.42}$$

where $\mathbf{B}$ and $\mathbf{A}$ are linear operators that can be written as

$$\mathbf{B} = I + \alpha A \kappa \begin{pmatrix} -\alpha\delta_+ & -\delta_- \\ \delta_+ & \alpha\delta_- \end{pmatrix}, \tag{4.43}$$

$$\mathbf{A} = \kappa d \begin{pmatrix} \Delta & -\alpha\Delta \\ -\alpha\Delta & \Delta \end{pmatrix} + \nu \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4.44}$$

We now detail the system of equations to solve during pseudo-arclength continuation. The dot notation represents differentiation with respect to arclength s. For a simplified notation define x and dx as follows:

$$x := (\phi_1, \phi_2, c)^T, \quad dx := (d\phi_1, d\phi_2, dc)^T. \tag{4.45}$$

The block matrix form of the equations take the following form:

$$\begin{bmatrix} f_x & f_\lambda \\ (\dot{x}^{(0)}_{j+1})^T & (\dot{\lambda}^{(0)}_{j+1})^T \end{bmatrix}_{(x^{(k)}_{j+1},\lambda^{(k)}_{j+1})} \begin{bmatrix} dx \\ d\lambda \end{bmatrix} = \begin{bmatrix} f(x^{(k)}_{j+1},\lambda^{(k)}_{j+1}) \\ (\dot{x}^{(0)}_{j+1})^T(x^{(k)}_{j+1}-x^{(0)}_{j+1})+(\dot{\lambda}^{(0)}_{j+1})^T(\lambda^{(k)}_{j+1}-\lambda^{(0)}_{j+1})-\Delta_s \end{bmatrix} \tag{4.46}$$

where $f=[G,\sigma]^T$ for a phase condition σ, the parameter λ represents either α or ν, and Δ_s is the arclength of the continuation step. In this setup, f_x and f_λ are in the following form:

$$f_x = \left[\begin{array}{cc|c} D_{\phi_1}G & D_{\phi_2}G & D_cG \\ \hline D_{\phi_1}\sigma & \vec{0} & 0 \end{array}\right] \quad f_\lambda = \left[\begin{array}{c} D_\lambda G \\ \hline 0 \end{array}\right] \tag{4.47}$$

The initial tangent vector may be chosen by selecting an element in the nullspace of $[f_x\ f_\lambda]$ since the tangent vector $[\dot{x}\ \dot{\lambda}]$ satisfies the following equation via the chain rule:

$$f_x\,\dot{x}+f_\lambda\,\dot{\lambda}=0. \tag{4.48}$$

Our previous analysis indicated that a vector of the form $w=(0,\phi_2',0)^T$ is in the kernel of f_x. As a result, the choice of initial tangent vectors emanating from the saddle-node bifurcation may be selected from a null space Matlab call $\texttt{null}([\texttt{f}_\texttt{x}\ \texttt{f}_\lambda])$ and adding multiples of w to traverse forward and backward. Subsequent vectors are chosen by a secant method

$$\begin{bmatrix} x^{(k)}_{j+1}-x^{(0)}_{j+1} \\ \lambda^{(k)}_{j+1}-\lambda^{(0)}_{j+1} \end{bmatrix} /\| \begin{bmatrix} x^{(k)}_{j+1}-x^{(0)}_{j+1} \\ \lambda^{(k)}_{j+1}-\lambda^{(0)}_{j+1} \end{bmatrix} \|. \tag{4.49}$$

4.7 Numerical Results

The uncoupled pair of traveling waves from which the continuation is started has a depiction in Figure 4.1. The saddle-node bifurcation as detailed in Chapter 3 appears to be a small effect as it is at the error resolution of our solver. In addition, perhaps the two waves merge before $\alpha = .01$. Below, we have examined the effects of both coupling parameters and the role of the parameter A on the shape of the profiles and the effect on the wave speed.

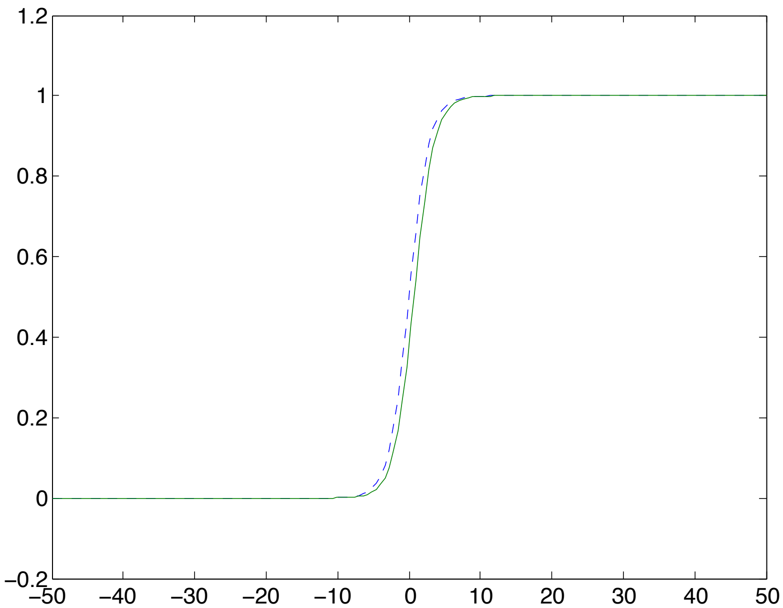


Figure 4.1: Depiction of pair traveling wave profiles. The profiles are offset by $\beta = 1$.

The following results detailed below are for the following parameter values: $a = .80$, $A = .25$, $d = 1$. The phase delay β was initially set 1 to put the system in the saddle-node bifurcation regime. To assess the impact of continuation in α, ν was set to zero, and path-following was as α varied from 0 to .95. The results are shown in Table 4.3. These decreasing wave speed values correspond to the phenomenon observed in

[11], [41] where the wave speed decreases by a factor of $\sqrt{1+\alpha}$ in parameter regime away from propagation failure.

α	0	.2	.4	.6	.8
c	0.4212	0.3845	0.3559	0.3329	0.3139

Table 4.3: Wave Speed Values for $a = .80$ and $A = .25$.

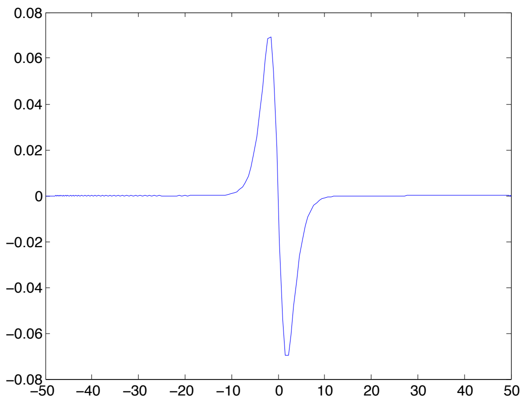
To depict the change in the wave profiles, the difference of wave profiles are plotted in Figure 4.2. In the upper left subfigure, $\phi_1|_{\alpha=0} - \phi_1|_{\alpha=0.80}$ is depicted. This plot suggests that ϕ_1 gains a scaled multiple of $\Delta\phi_1$ as α is increased. In the upper right subfigure, $\phi_2|_{\alpha=0} - \phi_1|_{\alpha=0.80}$ is plotted. This plot suggests that not only does ϕ_2 add a scaled multiple of $\Delta\phi_2$ but also a scaled multiple ϕ_2' for translation. The bottom two subfigures depict the difference in the two profiles at $\alpha = 0$ and $\alpha = .80$, respectively. Both of these figures suggest that ϕ_1 and ϕ_2 differ only by translation.

As the parameter A varies, the wave speed varies. While not a large effect, as A increases the wave speed increases as in Table 4.4. The second profile, ϕ_2 translates less as A increases from .25 to .45. Figure 4.3 depicts $\phi_2|_{\alpha=0} - \phi_2|_{\alpha=.80}$ for both $A = .25$ and $A = .45$.

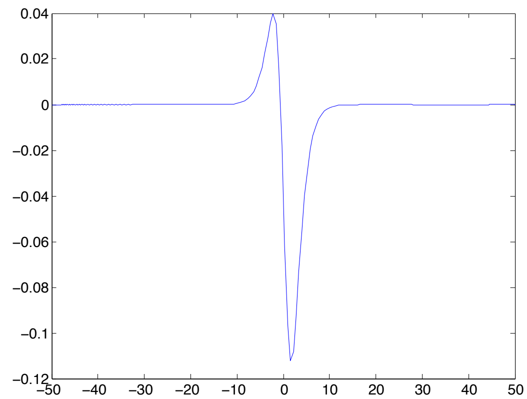
A	α	0	.2	.4	.6	.8
.25	c	0.4212	0.3845	0.3559	0.3329	0.3139
.35	c	0.4212	0.3847	0.3563	0.3334	0.3145
.45	c	0.4212	0.3848	0.3565	0.3337	0.3149

Table 4.4: Wave Speed Values for $a = .80$ and α and A varying.

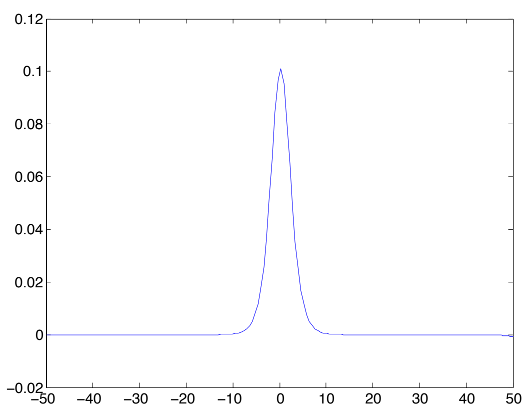
As the second coupling parameter is varied, differing behavior is exhibited depending on whether ν is positive or negative. When $\nu < 0$, then first profile begins to lose monotonicity as depicted in Figure 4.4. We also find that the wave speed increases from .3754 to .3842 as ν varies from 0 to -0.0118. When ν is positive, the continuation process is rapidly halted due to error tolerances being exceeded.



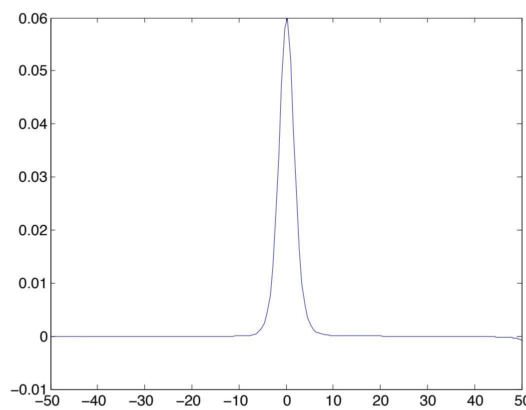
(a) $\phi_1|_{\alpha=0} - \phi_1|_{\alpha=.80}$



(b) $\phi_2|_{\alpha=0} - \phi_2|_{\alpha=.80}$



(c) $\phi_1|_{\alpha=0} - \phi_2|_{\alpha=0}$



(d) $\phi_1|_{\alpha=.80} - \phi_2|_{\alpha=.80}$

Figure 4.2: Profile Changes as α is varied between 0 and .80.

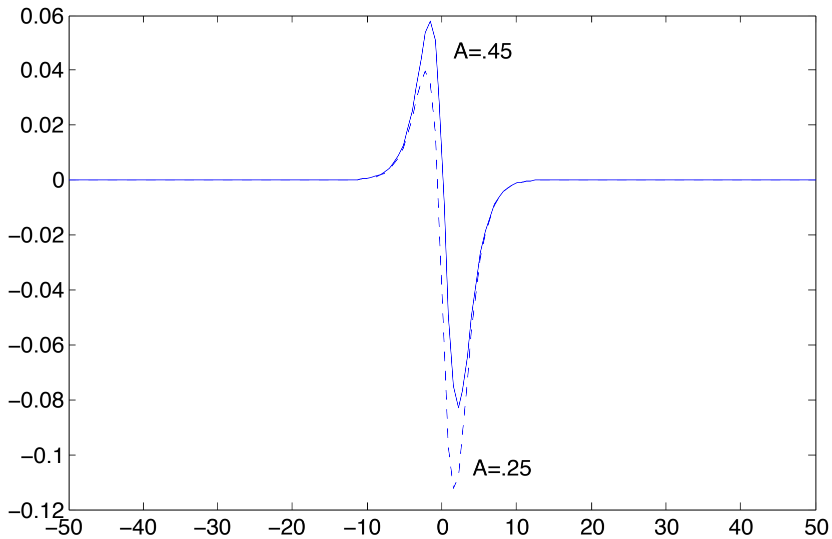


Figure 4.3: Profile changes as as α is varied between 0 and .80 for both $A = .25$ and $A = .45$.

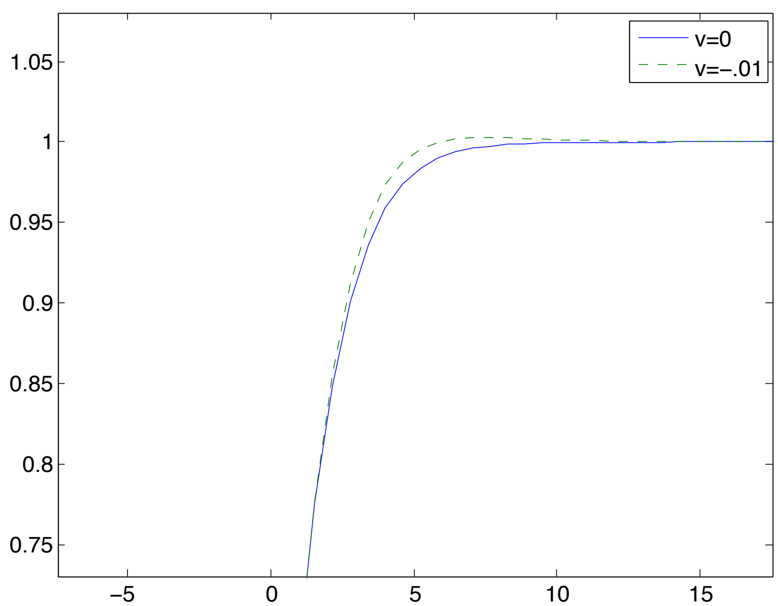


Figure 4.4: The first profile ϕ_1 begins to lose monotonicity as ν is decreased from 0 to $-.01$ for fixed $\alpha = .25$.

Chapter 5

Future Work

5.1 Other Truncation Strategies

5.1.1 Projection Boundary Conditions

Assuming that the interval is truncated to $[T_-, T_+]$, the idea of projection boundary conditions [10] is to enforce that the solutions lies in the appropriate subspaces at T_- and T_+. If $A_\pm$ is the linearization about equilibria at $\pm\infty$, then the projection boundary conditions in the case of $+\infty$ require that the difference $\phi(T_+) - \phi(+\infty)$ is orthogonal to the unstable subspace of A_+. This orthogonality ensures that the solution ϕ approaches the right boundary condition via the stable manifold at $+\infty$. A similar condition is imposed at $-\infty$. The application of this idea toward mixed type problems requires modification. Since the unstable and stable subspaces are infinite for mixed type problems, work would be necessary to adapt the idea of projection boundary conditions.

5.1.2 PML

While the context is different, the idea of perfectly matched layers (pml) is worth consideration. In [9] Berenger developed the method of perfectly matched layers, and it has been used for linear wave equations. The method imparts decay in the solution where decay is absent. In essence, the idea of perfectly matched layers is to take an unbounded domain $x \in (-\infty, +\infty)$ and analytically continue the solution/equation into the complex domain (notation $\tilde{x}$). The equations can be rewritten in terms of the real part of $\tilde{x}$ by making the change of variables

$$\tilde{x} = x + if(x), \qquad x \in \mathbb{R}. \tag{5.1}$$

After the solution is continued into the complex plane, the unbounded domain can be truncated within the region where $f(x) \neq 0$ (the absorbing region).

By making the above assumptions, the solution $u(x,t)$ solves a linear homogeneous problem far from the region of interest, and takes the form of a superposition of plane waves

$$u(x,t) = \sum_{k,\omega} a_{k,\omega} e^{i(kx-\omega t)}, \tag{5.2}$$

noting that the ratio $c = \frac{\omega}{k}$ of the frequency ω and the inverse of the wavelength k is a constant called the phase velocity of the wave. The exponential function is smooth, so the solution $u(x,t)$ can be analytically continued to $u(\tilde{x},t)$ for complex $\tilde{x}$. The choice of $f(x)$ is key to ensure that u experiences exponential spatial decay. For example set $f(x) = \int^x H(t-100)dt$, where $H(x)$ is the Heaviside function, then a plane wave takes the following form

$$\begin{aligned} e^{i(k\tilde{x}-\omega t)} &= e^{i(kx-\omega t)} e^{-kf(x)} \\ &= \begin{cases} e^{i(kx-\omega t)}, & \text{for } x < 100 \\ e^{i(kx-\omega t)} e^{-k(x-100)}, & \text{for } x > 100. \end{cases} \end{aligned} \tag{5.3}$$

The above wave will experience exponential decay for $x > 100$ for k positive. Decay is similarly introduced in the $-x$ direction.

The typical formulation for $f(x)$ is in practice slightly different by incorporating the frequency ω, where

$$\frac{df}{dx} = \frac{\sigma(x)}{\omega}. \tag{5.4}$$

In this manner the decay factor for a wave traveling toward $+\infty$, for example, is independent of k and ω:

$$e^{-kf(x)} = e^{\frac{-k}{w}\int^{x}\sigma(t)dt} = e^{\frac{-1}{c}(x-100)}. \tag{5.5}$$

The unbounded interval can now be truncated to $[T_-, T_+]$ in the region where $f(x) \neq 0$ using Dirichlet boundary conditions. Whereas the unmodified wave equation with Dirichlet boundary conditions reflects traveling waves, the pml approach will only reflect attentuated traveling wave (by a factor of $e^{\frac{-1}{c}(x-100)}$). In this sense, the method is ”reflection-less,” and the region of interest and absorbing layer as said to be perfectly matched.

5.2 Higher Dimensional Systems

The idea of higher dimensional systems that decouple into d identical copies of the same equation can be extended to the bundling of the parallel nerve fibers. In nature, parallel nerve fibers can coexist in a bundled arrangement, and as a first step in modeling a hexagonal arrangement of the fibers may be considered. This model would, by examining a cross-sectional slice, have fibers at each of the vertices of the hexagon and one additional fiber in the center of the polygon. Chapter 2 suggests that this model with seven coupled fibers will admit a linear operator with a kernel of dimension six. A number of variations of this project are present:

- How should the coupling be chosen between the fibers in the bundle?
- Which solutions are stable?
- What about symmetry and antisymmetry?

5.3 Neutral Delay Equations

In [20], [4] neutral delay equations are considered with a single delay. There are a few issues specific to neutral equations that we detail below. First and foremost is the issue of discontinuities in the derivative terms. One must choose whether to track the discontinuities and appropriately place mesh points at the discontinuities. For constant shifts this entails the solver being cognizant of the shift values, and appropriately propagating them along the real line. A more difficult issue is if the shifts are state dependent, and this issue is not discussed in the above references.

Related to the issue of the discontinuities is the interpolation of $\dot{x}(t-\tau)$ as in the following equation [4]

$$\dot{x}(t) = f(x(t), x(t-\tau), \dot{x}(t-\tau)). \tag{5.6}$$

If τ is not an integer multiple of h, where h is mesh size, then interpolation will have to be performed to evaluate $\dot{x}(t-\tau)$ in addition to $x(t-\tau)$. An alternative formulation to the neutral problem in [4] provides direct access to the derivative at collocation points. The extended neutral collocation (ENC) identifies $y(t) = \dot{x}(t)$ and then solves the extended system

$$\begin{aligned} y(t) &= \dot{x}(t) \\ y(t) &= f(x(t), x(t-\tau), y(t-\tau)). \end{aligned} \tag{5.7}$$

One last issue that has been discussed before, is the relative sizes of the coefficients of the derivative terms. If the coefficients are not restricted in some manner the neutral system may have discontinuities in the solution, not just the derivatives. In [20], restrictions are placed on d in the equation

$$\frac{d}{dt}\{y(t) + dy(t-\tau)\} = a(t)y(t) + f(t). \tag{5.8}$$

The coefficient d must remain small relative to 1, the coefficient of $y(t)$.

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